Noncommutative Involutive Bases

Thesis submitted to the University of Wales in support of the application for the degree of Philosophiæ Doctor

by

Gareth Alun Evans

School of Informatics
The University of Wales, Bangor
September 2005

DECLARATION

This work has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree.

Signed .... (candidate)
Date ....

STATEMENT 1

This thesis is the result of my own investigations, except where otherwise stated.
Other sources are acknowledged by explicit references. A bibliography is appended.

Signed .... (candidate)
Date ....

STATEMENT 2

I hereby give consent for my thesis, if accepted, to be available for photocopying and for inter-library loan, and for the title and summary to be made available to outside organisations.

Signed .... (candidate)
Date ....

Summary

The theory of Gröbner Bases originated in the work of Buchberger [11] and is now considered to be one of the most important and useful areas of symbolic computation. A great deal of effort has been put into improving Buchberger's algorithm for computing a Gröbner Basis, and indeed in finding alternative methods of computing Gröbner Bases. Two of these methods include the Gröbner Walk method [1] and the computation of Involutive Bases [58].

By the mid 1980's, Buchberger's work had been generalised for noncommutative polynomial rings by Bergman [8] and Mora [45]. This thesis provides the corresponding generalisation for Involutive Bases and (to a lesser extent) the Gröbner Walk, with the main results being as follows.

(1) Algorithms for several new noncommutative involutive divisions are given, including strong; weak; global and local divisions.

(2) An algorithm for computing a noncommutative Involutive Basis is given. When used with one of the aforementioned involutive divisions, it is shown that this algorithm returns a noncommutative Gröbner Basis on termination.

(3) An algorithm for a noncommutative Gröbner Walk is given, in the case of conversion between two harmonious monomial orderings. It is shown that this algorithm generalises to give an algorithm for performing a noncommutative Involutive Walk, again in the case of conversion between two harmonious monomial orderings.

(4) Two new properties of commutative involutive divisions are introduced (stability and extendibility), respectively ensuring the termination of the Involutive Basis algorithm and the applicability (under certain conditions) of homogeneous methods of computing Involutive Bases.

Source code for an initial implementation of an algorithm to compute noncommutative Involutive Bases is provided in Appendix B. This source code, written using ANSI C and a series of libraries (AlgLib) provided by MSSRC [46], forms part of a larger collection of programs providing examples for the thesis, including implementations of the commutative and noncommutative Gröbner Basis algorithms [11, 45]; the commutative Involutive Basis algorithm for the Pommaret and Janet involutive divisions [58]; and the Knuth-Bendix critical pairs completion algorithm for monoid rewrite systems [39].

Acknowledgements

Many people have inspired me to complete this thesis, and I would like to take this opportunity to thank some of them now.

I would like to start by thanking my family for their constant support, especially my parents who have encouraged me every step of the way. *Mae fy nyled yn fawr iawn i chi.*

I would like to thank Prof. Larry Lambe from MSSRC, whose software allowed me to test my theories in a way that would not have been possible elsewhere.

Thanks to all the Mathematics Staff and Students I have had the pleasure of working with over the past seven years. Particular thanks go to Dr. Bryn Davies, who encouraged me to think independently; to Dr. Jan Abas, who inspired me to reach goals I never thought I could reach; and to Prof. Ronnie Brown, who introduced me to Involutive Bases.

I would like to finish by thanking my Supervisor Dr. Chris Wensley. Our regular meetings kept the cogs in motion and his insightful comments enabled me to avoid wrong turnings and to get the little details right. *Diolch yn fawr!*

This work has been gratefully supported by the EPSRC and by the School of Informatics at the University of Wales, Bangor.

Typeset using LaTeX, XFig and XY-pic.

"No one has ever done anything like this."
"That's why it's going to work."

The Matrix [54]

Contents

List of Algorithms

Introduction

Background

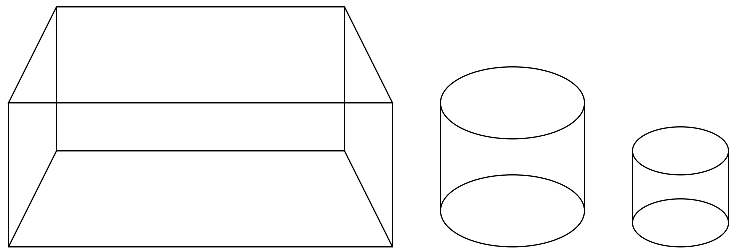
Gröbner Bases

During the second half of the twentieth century, one of the most successful applications of symbolic computation was in the development and application of *Gröbner Basis* theory for finding special bases of ideals in commutative polynomials rings. Pioneered by Bruno Buchberger in 1965 [11], the theory allowed an answer to the question "What is the unique remainder when a polynomial is divided by a set of polynomials?". Buchberger's algorithm for computing a Gröbner Basis was improved and refined over several decades [1, 10, 21, 29], aided by the development of powerful symbolic computation systems over the same period. Today there is an implementation of Buchberger's algorithm in virtually all general purpose symbolic computation systems, including Maple [55] and Mathematica [57], and many more specialised systems.

What is a Gröbner Basis?

Consider the problem of finding the remainder when a number is divided by a set of numbers. If the dividing set contains just one number, then the problem only has one solution. For example, "5" is the only possible answer to the question "What is $20 \div 4$?". If the dividing set contains more than one number however, there may be several solutions, as the division can potentially be performed in more than one way.

Example. Consider a tank containing 21L of water. Given two empty jugs, one with a capacity of 2L and the other 5L, is it possible to empty the tank using just the jugs, assuming only full jugs of water may be removed from the tank?



Trying to empty the tank using the 2L jug only, we are able to remove $10 \times 2 = 20$L of water from the tank, and we are left with 1L of water in the tank. Repeating with the 5L jug, we are again left with 1L of water in the tank. If we alternate between the jugs however (removing 2L of water followed by 5L followed by 2L and so on), the tank this time does become empty, because $21 = 2 + 5 + 2 + 5 + 2 + 5$.

The observation that we are left with a different volume of water in the tank dependent upon how we try to empty it corresponds to the idea that the remainder obtained when dividing the number 21 by the numbers 2 and 5 is dependent upon how the division is performed.

This idea also applies when dividing polynomials by sets of polynomials — remainders here will also be dependent upon how the division is performed. However, if we divide a polynomial with respect to a set of polynomials that is a Gröbner Basis, then we will always obtain the same remainder no matter how the division is performed. This fact, along with the fact that any set of polynomials can be transformed into an equivalent set of polynomials that is a Gröbner Basis, provides the main ingredients of Gröbner Basis theory.

Remark. The 'Gröbner Basis' for our water tank example would be just a 1L jug, allowing us to empty any tank containing nL of water (where $n \in \mathbb{N}$).

Applications

There are numerous applications of Gröbner Bases in all branches of mathematics, computer science, physics and engineering [12]. Topics vary from geometric theorem proving to solving systems of polynomial equations, and from algebraic coding theory to the design of experiments in statistics.

Example. Let $F := \{x + y + z = 6,\ x^2 + y^2 + z^2 = 14,\ x^3 + y^3 + z^3 = 36\}$ be a set of polynomial equations. One way of solving this set for x, y and z is to compute a *lexicographic* Gröbner Basis for F. This yields the set $G := \{x+y+z = 6,\ y^2+yz+z^2-6y-6z = -11,\ z^3-6z^2+11z = 6\}$, the final member of which is a univariate polynomial in z, a polynomial we can solve to deduce that $z = 1$, 2 or 3. Substituting back into the second member of G, when $z = 1$, we obtain the polynomial $y^2-5y+6 = 0$, which enables us to deduce that $y = 2$ or 3; when $z = 2$, we obtain the polynomial $y^2 - 4y + 3 = 0$, which enables us to deduce that $y = 1$ or 3; and when $z = 3$, we obtain the polynomial $y^2 - 3y + 2 = 0$, which enables us to deduce that $y = 1$ or 2. Further substitution into $x+y+z = 6$ then enables us to deduce the value of x in each of the above cases, enabling us to give the following table of solutions for F.

x	3	2	3	1	2	1
y	2	3	1	3	1	2
z	1	1	2	2	3	3

Involutive Bases

As Gröbner Bases became popular, researchers noticed a connection between Buchberger's ideas and ideas originating from the Janet-Riquier theory of Partial Differential Equations developed in the early 20th century (see for example [44]). This link was completed for commutative polynomial rings by Zharkov and Blinkov in the early 1990's [58] when they gave an algorithm to compute an *Involutive Basis* that provides an alternative way of computing a Gröbner Basis. Early implementations of this algorithm (an elementary introduction to which can be found in [13]) compared favourably with the most advanced implementations of Buchberger's algorithm, with results in [25] showing the potential of the Involutive method in terms of efficiency.

What is an Involutive Basis?

Given a Gröbner Basis G, we know that the remainder obtained from dividing a polynomial with respect to G will always be the same no matter how the division is performed. With an Involutive Basis, the difference is that there is only one way for the division to be performed, so that unique remainders are also obtained uniquely.

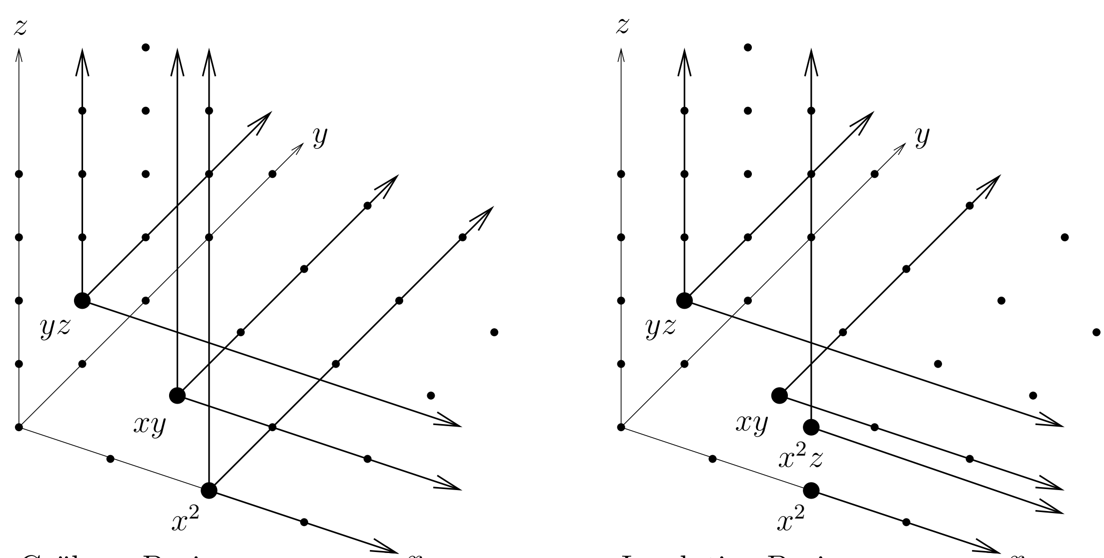
This effect is achieved through assigning a set of *multiplicative variables* to each polynomial

in an Involutive Basis H, imposing a restriction on how polynomials may be divided by H by only allowing any polynomial $h \in H$ to be multiplied by its corresponding multiplicative variables. Popular schemes of assigning multiplicative variables include those based on the work of Janet [35], Thomas [52] and Pommaret [47].

Example. Consider the Janet Involutive Basis $H := \{xy - z,\ yz + 2x + z,\ 2x^2 + xz + z^2,\ 2x^2z + xz^2 + z^3\}$ with multiplicative variables as shown in the table below.

Polynomial	Janet Multiplicative Variables
$xy - z$	$\{x, y\}$
$yz + 2x + z$	$\{x, y, z\}$
$2x^2 + xz + z^2$	$\{x\}$
$2x^2z + xz^2 + z^3$	$\{x, z\}$

To illustrate that any polynomial may only be *involutively divisible* by at most one member of any Involutive Basis, we include the following two diagrams, showing which monomials are involutively divisible by H, and which are divisible by the corresponding Gröbner Basis $G := \{xy - z,\ yz + 2x + z,\ 2x^2 + xz + z^2\}$.



Gröbner Basis   Involutive Basis

Note that the irreducible monomials of both bases all appear in the set $\{1,\ x,\ y^i,\ z^i,\ xz^i\}$, where $i \geqslant 1$; and that the cube, the 2 planes and the line shown in the right hand diagram do not overlap.

Noncommutative Bases

There are certain types of noncommutative algebra to which methods for commutative Gröbner Bases may be applied. Typically, these are algebras with generators $\{x_1, \ldots, x_n\}$ for which products $x_j x_i$ with $j > i$ may be rewritten as ($x_i x_j +$ other terms). For example, version 3-0-0 of Singular [31] (released in June 2005) allows the computation of Gröbner Bases for G-algebras.

To compute Gröbner Bases for ideals in free associative algebras however, one must turn to the theory of *noncommutative Gröbner Bases*. Based on the work of Bergman [8] and Mora [45], the theory answers the question "What is the remainder when a noncommutative polynomial is divided by a set of noncommutative polynomials?", and allows us to find Gröbner Bases for such algebras as path algebras [37].

The final piece of the jigsaw is to mirror the application of Zharkov and Blinkov's Involutive methods to the noncommutative case. This thesis provides the first extended attempt at accomplishing this task, improving the author's first basic algorithms for computing *noncommutative Involutive Bases* [20] and providing a full theoretical foundation for these algorithms.

Structure and Principal Results

This thesis can be broadly divided into two parts: Chapters 1 through 4 survey the building blocks required for the theory of noncommutative Involutive Bases; the remainder of the thesis then describes this theory together with different ways of computing noncommutative Involutive Bases.

Part 1

Chapter 1 contains accounts of some necessary preliminaries for our studies – a review of both commutative and noncommutative polynomial rings; ideals; monomial orderings; and polynomial division.

We survey the theory of *commutative Gröbner Bases* in Chapter 2, basing our account on many sources, but mainly on the books [7] and [22]. We present the theory from the viewpoint of S-polynomials (for example defining a Gröbner Basis in terms of S-

polynomials), mainly because Buchberger's algorithm for computing a Gröbner Basis deals predominantly with S-polynomials. Towards the end of the Chapter, we describe some of the theoretical improvements of Buchberger's algorithm, including the usage of selection strategies, optimal variable orderings and Logged Gröbner Bases.

The viewpoint of defining Gröbner Bases in terms of S-polynomials continues in Chapter 3, where we encounter the theory of *noncommutative Gröbner Bases*. We discover that the theory is quite similar to that found in the previous chapter, apart from the definition of an S-polynomial and the fact that not all input bases will have finite Gröbner Bases.

In Chapter 4, we acquaint ourselves with the theory of *commutative Involutive Bases*. This is based on the work of Zharkov and Blinkov [58]; Gerdt and Blinkov [25, 26]; Gerdt [23, 24]; Seiler [50, 51]; and Apel [2, 3], with the notation and conventions taken from a combination of these papers. For example, notation for involutive cones and multiplicative variables is taken from [25], and the definition of an involutive division and the algorithm for computing an Involutive Basis is taken from [50].

As for the content of Chapter 4, we introduce the Janet, Pommaret and Thomas divisions in Section 4.1; describe what is meant by a prolongation and autoreduction in Section 4.2; introduce the properties of continuity and constructivity in Section 4.3; give the Involutive Basis algorithm in Section 4.4; and describe some improvements to this algorithm in Section 4.5. In between all of this, we introduce two new properties of involutive divisions, stability and extendibility, that ensure (respectively) the termination of the Involutive Basis algorithm and the applicability (under certain conditions) of homogeneous methods of computing Involutive Bases.

Part 2

The main results of the thesis are contained in Chapter 5, where we introduce the theory of *noncommutative Involutive Bases*. In Section 5.1, we define two methods of performing noncommutative involutive reduction, the first of which (using thin divisors) allows the mirroring of theory from Chapter 4, and the second of which (using thick divisors) allows efficient computation of involutive remainders. We also define what is meant by a noncommutative involutive division, and give an algorithm for performing noncommutative involutive reduction.

In Section 5.2, we generalise the notions of prolongation and autoreduction to the non-

commutative case, introducing two different types of prolongation (left and right) to reflect the fact that left and right multiplication are different operations in noncommutative polynomial rings. These notions are then utilised in the algorithm for computing a noncommutative Involutive Basis, which we present in Section 5.3.

In Section 5.4, we introduce two properties of noncommutative involutive divisions. Continuity helps ensure that any Locally Involutive Basis is an Involutive Basis; conclusivity ensures that for any given input basis, a finite Involutive Basis will exist if and only if a finite Gröbner Basis exists. A third property is also introduced for weak involutive divisions to ensure that any Locally Involutive Basis is a Gröbner Basis (Involutive Bases with respect to strong involutive divisions are automatically Gröbner Bases).

Section 5.5 provides several involutive divisions for use with the noncommutative Involutive Basis algorithm, including two global divisions and ten local divisions. The properties of these divisions are analysed, with full proofs given that certain divisions satisfy certain properties. We also show that some divisions are naturally suited for efficient involutive reduction, and speculate on the existence of further involutive divisions.

In Section 5.6, we briefly discuss the topic of the termination of the noncommutative Involutive Basis algorithm. In Section 5.7, we provide several examples showing how noncommutative Involutive Bases are computed, including examples demonstrating the computation of involutive complete rewrite systems for groups. Finally, in Section 5.8, we discuss improvements to the noncommutative Involutive Basis algorithm, including how to introduce efficient involutive reduction and Logged Involutive Bases.

Chapter 6 introduces and generalises the theory of the *Gröbner Walk*, where a Gröbner Basis with respect to one monomial ordering may be computed from a Gröbner Basis with respect to another monomial ordering. In Section 6.1, we summarise the theory of the commutative Gröbner Walk (based on the papers [1] and [18]), and we describe a generalisation of the theory to the Involutive case due to Golubitsky [30]. In Section 6.2, we then go on to partially generalise the theory to the noncommutative case, giving algorithms to perform both Gröbner and Involutive Walks between two harmonious monomial orderings.

After some concluding remarks in Chapter 7, we provide full proofs for two Propositions from Section 5.5 in Appendix A. Appendix B then provides ANSI C source code for an initial implementation of the noncommutative Involutive Basis algorithm, together with

a brief description of the AlgLib libraries used in conjunction with the code. Finally, in Appendix C, we provide sample sessions showing the program given in Appendix B in action.

Chapter 1

Preliminaries

In this chapter, we will set out some algebraic concepts that will be used extensively in the following chapters. In particular, we will introduce polynomial rings and ideals, the main objects of study in this thesis.

1.1 Rings and Ideals

1.1.1 Groups and Rings

Definition 1.1.1 A *binary operation* on a set S is a function $* : S \times S \to S$ such that associated with each ordered pair (a, b) of elements of S is a uniquely defined element $(a * b) \in S$.

Definition 1.1.2 A *group* is a set G, with a binary operation $*$, such that the following conditions hold.

(a) $g_1 * g_2 \in G$ for all $g_1, g_2 \in G$ (closure).

(b) $g_1 * (g_2 * g_3) = (g_1 * g_2) * g_3$ for all $g_1, g_2, g_3 \in G$ (associativity).

(c) There exists an element $e \in G$ such that for all $g \in G$, $e * g = g = g * e$ (identity).

(d) For each element $g \in G$, there exists an element $g^{-1} \in G$ such that $g^{-1} * g = e = g * g^{-1}$ (inverses).

Definition 1.1.3 A group G is *abelian* if the binary operation of the group is commutative, that is $g_1 * g_2 = g_2 * g_1$ for all $g_1, g_2 \in G$. The operation in an abelian group is often written additively, as $g_1 + g_2$, with the inverse of g written $-g$.

Definition 1.1.4 A *rng* is a set R with two binary operations $+$ and $\times$, known as addition and multiplication, such that addition has an identity element 0, called *zero*, and the following axioms hold.

(a) R is an abelian group with respect to addition.

(b) $(r_1 \times r_2) \times r_3 = r_1 \times (r_2 \times r_3)$ for all $r_1, r_2, r_3 \in R$ (multiplication is associative).

(c) $r_1 \times (r_2 + r_3) = r_1 \times r_2 + r_1 \times r_3$ and $(r_1 + r_2) \times r_3 = r_1 \times r_3 + r_2 \times r_3$ for all $r_1, r_2, r_3 \in R$ (the distributive laws hold).

Definition 1.1.5 A rng R is a *ring* if it contains a unique element 1, called the *unit* element, such that $1 \neq 0$ and $1 \times r = r = r \times 1$ for all $r \in R$.

Definition 1.1.6 A ring R is *commutative* if multiplication (as well as addition) is commutative, that is $r_1 \times r_2 = r_2 \times r_1$ for all $r_1, r_2 \in R$.

Definition 1.1.7 A ring R is *noncommutative* if $r_1 \times r_2 \neq r_2 \times r_1$ for some $r_1, r_2 \in R$.

Definition 1.1.8 If S is a subset of a ring R that is itself a ring under the same binary operations of addition and multiplication, then S is a *subring* of R.

Definition 1.1.9 A ring R is a *division ring* if every nonzero element $r \in R$ has a multiplicative inverse r^{-1}. A *field* is a commutative division ring.

1.1.2 Polynomial Rings

Commutative Polynomial Rings

A nontrivial *polynomial* p in n (commuting) variables $x_1, \ldots, x_n$ is usually written as a sum

$$p = \sum_{i=1}^{k} a_i x_1^{e_i^1} x_2^{e_i^2} \ldots x_n^{e_i^n}, \tag{1.1}$$

where k is a positive integer and each summand is a *term* made up of a nonzero *coefficient* a_i from some ring R and a *monomial* $x_1^{e_i^1} x_2^{e_i^2} \ldots x_n^{e_i^n}$ in which the exponents $e_i^1, \ldots, e_i^n$ are

nonnegative integers. It is clear that each monomial may be represented in terms of its exponents only, as a *multidegree* $e_i = (e_i^1, e_i^2, \ldots, e_i^n)$, so that a monomial may be written as a multiset $\mathbf{x}^{e_i}$ over the set $\{x_1, \ldots, x_n\}$. This leads to a more elegant representation of a nontrivial polynomial,

$$p = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \mathbf{x}^\alpha, \tag{1.2}$$

and we may think of such a polynomial as a function f from the set of all multidegrees $\mathbb{N}^n$ to the ring R with finite support (only a finite number of nonzero images).

Example 1.1.10 Let $p = 4x^2y + 2x + \frac{19}{80}$ be a polynomial in two variables x and y with coefficients in $\mathbb{Q}$. This polynomial can be represented by the function $f : \mathbb{N}^2 \to \mathbb{Q}$ given by

$$f(\alpha) = \begin{cases} 4, & \alpha = (2,1) \\ 2, & \alpha = (1,0) \\ \frac{19}{80}, & \alpha = (0,0) \\ 0 & \text{otherwise.} \end{cases}$$

Remark 1.1.11 The zero polynomial $p = 0$ is represented by the function $f(\alpha) = 0_R$ for all possible α. The constant polynomial $p = 1$ is represented by the function $f(\alpha) = 1_R$ for $\alpha = (0, 0, \ldots, 0)$, and $f(\alpha) = 0_R$ otherwise.

Remark 1.1.12 The product $m_1 \times m_2$ of two monomials m_1, m_2 with corresponding multidegrees $e_1, e_2 \in \mathbb{N}^n$ is the monomial corresponding to the multidegree $e_1 + e_2$. For example, if $m_1 = x_1^2x_2x_3^3$ and $m_2 = x_1x_2x_3^2$ (so that $e_1 = (2, 1, 3)$ and $e_2 = (1, 1, 2)$), then $m_1 \times m_2 = x_1^3x_2^2x_3^5$ as $e_1 + e_2 = (3, 2, 5)$.

Definition 1.1.13 Let $R[x_1, x_2, \ldots, x_n]$ denote the set of all functions $f : \mathbb{N}^n \to R$ such that each function f represents a polynomial in n variables $x_1, \ldots, x_n$ with coefficients over a ring R. Given two functions $f, g \in R[x_1, x_2, \ldots, x_n]$, let us define the functions $f + g$ and $f \times g$ as follows.

$$\begin{aligned} (f+g)(\alpha) &= f(\alpha) + g(\alpha) && \text{for all } \alpha \in \mathbb{N}^n; \\ (f \times g)(\alpha) &= \sum_{\beta+\gamma=\alpha} f(\beta) \times g(\gamma) && \text{for all } \alpha \in \mathbb{N}^n. \end{aligned}$$

Then the set $R[x_1, x_2, \ldots, x_n]$ becomes a ring, known as the *polynomial ring in n variables over R*, with the functions corresponding to the zero and constant polynomials being the respective zero and unit elements of the ring.

Remark 1.1.14 In $R[x_1, x_2, \ldots, x_n]$, R is known as the *coefficient ring*.

Noncommutative Polynomial Rings

A nontrivial *polynomial* p in n noncommuting variables $x_1, \ldots, x_n$ is usually written as a sum

$$p = \sum_{i=1}^{k} a_i w_i, \tag{1.3}$$

where k is a positive integer and each summand is a *term* made up of a nonzero *coefficient* a_i from some ring R and a *monomial* w_i that is a word over the alphabet $X = \{x_1, x_2, \ldots, x_n\}$. We may think of a noncommutative polynomial as a function f from the set of all words X^* to the ring R.

Remark 1.1.15 The zero polynomial $p = 0$ is the polynomial $0_R\varepsilon$, where ε is the empty word in X^*. Similarly $1_R\varepsilon$ is the constant polynomial $p = 1$.

Remark 1.1.16 The product $w_1 \times w_2$ of two monomials $w_1, w_2 \in X^*$ is given by concatenation. For example, if $X = \{x_1, x_2, x_3\}$, $w_1 = x_3^2 x_2$ and $w_2 = x_1^3 x_3$, then $w_1 \times w_2 = x_3^2 x_2 x_1^3 x_3$.

Definition 1.1.17 Let $R\langle x_1, x_2, \ldots, x_n\rangle$ denote the set of all functions $f : X^* \to R$ such that each function f represents a polynomial in n noncommuting variables with coefficients over a ring R. Given two functions $f, g \in R\langle x_1, x_2, \ldots, x_n\rangle$, let us define the functions $f + g$ and $f \times g$ as follows.

$$\begin{aligned} (f+g)(w) &= f(w) + g(w) && \text{for all } w \in X^*; \\ (f \times g)(w) &= \sum_{u \times v = w} f(u) \times g(v) && \text{for all } w \in X^*. \end{aligned}$$

Then the set $R\langle x_1, x_2, \ldots, x_n\rangle$ becomes a ring, known as the *noncommutative polynomial ring in n variables over R*, with the functions corresponding to the zero and constant polynomials being the respective zero and unit elements of the ring.

1.1.3 Ideals

Definition 1.1.18 Let $\mathcal{R}$ be an arbitrary commutative ring. An *ideal* J in $\mathcal{R}$ is a subring of $\mathcal{R}$ satisfying the following additional condition: $jr \in J$ for all $j \in J$, $r \in \mathcal{R}$.

Remark 1.1.19 In the above definition, if $\mathcal{R}$ is a polynomial ring in n variables over a ring R ($\mathcal{R} = R[x_1, \ldots, x_n]$), the ideal J is a *polynomial ideal.* We will only consider polynomial ideals in this thesis.

Definition 1.1.20 Let $\mathcal{R}$ be an arbitrary noncommutative ring.

- A *left (right) ideal* J in $\mathcal{R}$ is a subring of $\mathcal{R}$ satisfying the following additional condition: $rj \in J$ ($jr \in J$) for all $j \in J$, $r \in \mathcal{R}$.
- A *two-sided ideal* J in $\mathcal{R}$ is a subring of $\mathcal{R}$ satisfying the following additional condition: $r_1 j r_2 \in J$ for all $j \in J$, $r_1, r_2 \in \mathcal{R}$.

Remark 1.1.21 Unless otherwise stated, all noncommutative ideals considered in this thesis will be two-sided ideals.

Definition 1.1.22 A set of polynomials $P = \{p_1, p_2, \ldots, p_m\}$ is a *basis* for an ideal J of a noncommutative polynomial ring $\mathcal{R}$ if every polynomial $q \in J$ can be written as

$$q = \sum_{i=1}^{k} \ell_i p_i r_i \quad (\ell_i, r_i \in \mathcal{R},\ p_i \in P). \tag{1.4}$$

We say that P generates J, written $J = \langle P \rangle$.

Remark 1.1.23 The above definition has an obvious generalisation for left and right ideals of noncommutative polynomial rings and for ideals of commutative polynomial rings.

Example 1.1.24 Let $\mathcal{R}$ be the noncommutative polynomial ring $\mathbb{Q}\langle x, y\rangle$, and let $J = \langle P \rangle$ be an ideal in $\mathcal{R}$, where $P := \{x^2y + yx - 2,\ yxy - x + 4y\}$. Consider the polynomial $q := 2x^3y + yx^2y + 2xyx - 4x^2y + x^3 - 2xy - 4x$, and let us ask if q is a member of the ideal. To answer this question, we have to find out if there is an expression for q of the type shown in Equation (1.4). In this case, it turns out that q is indeed a member of the ideal (because $q = 2x(x^2y + yx - 2) + (x^2y + yx - 2)xy - x^2(yxy - x + 4y)$), but how would we answer the question in general? This problem is known as the Ideal Membership Problem and is stated as follows.

Definition 1.1.25 (The Ideal Membership Problem) Given an ideal J and a polynomial q, does $q \in J$?

As we shall see shortly, the Ideal Membership Problem can be solved by dividing a polynomial with respect to a Gröbner Basis for the ideal J. But before we can discuss this, we must first introduce the notion of polynomial division, for which we require a fixed ordering on the monomials in any given polynomial.

1.2 Monomial Orderings

A *monomial ordering* is a bivariate function O which tells us which monomial is the larger of any two given monomials m_1 and m_2. We will use the convention that $\mathrm{O}(m_1, m_2) = 1$ if and only if $m_1 < m_2$, and $\mathrm{O}(m_1, m_2) = 0$ if and only if $m_1 \geqslant m_2$. We can use a monomial ordering to order an arbitrary polynomial p by inducing an order on the terms of p from the order on the monomials associated with the terms.

Definition 1.2.1 A monomial ordering O is *admissible* if the following conditions are satisfied.

(a) $1 < m$ for all monomials $m \neq 1$.

(b) $m_1 < m_2 \Rightarrow m_\ell m_1 m_r < m_\ell m_2 m_r$ for all monomials[1] m_1, m_2, m_ℓ, m_r.

By convention, a polynomial is always written in descending order (with respect to a given monomial ordering), so that the *leading term* of the polynomial (with associated *leading coefficient* and *leading monomial*) always comes first.

Remark 1.2.2 For an arbitrary polynomial p, we will use $\mathrm{LT}(p)$, $\mathrm{LM}(p)$ and $\mathrm{LC}(p)$ to denote the leading term, leading monomial and leading coefficient of p respectively.

1.2.1 Commutative Monomial Orderings

A monomial ordering usually requires an ordering on the variables in our chosen polynomial ring. Given such a ring $R[x_1, x_2, \ldots, x_n]$, we will assume this order to be $x_1 > x_2 > \cdots > x_n$.

We shall now consider the most frequently used monomial orderings, where throughout m_1 and m_2 will denote arbitrary monomials (with associated multidegrees $e_1 = (e_1^1, e_1^2, \ldots, e_1^n)$

[1]For a commutative monomial ordering, we can ignore the monomial m_r.

and $e_2 = (e_2^1, e_2^2, \ldots, e_2^n))$, and $\deg(m_i)$ will denote the total degree of the monomial m_i (for example $\deg(x^2yz) = 4$). All orderings considered will be admissible.

The Lexicographical Ordering (Lex)

Define $m_1 < m_2$ if $e_1^i < e_2^i$ for some $1 \leqslant i \leqslant n$ and $e_1^j = e_2^j$ for all $1 \leqslant j < i$. In words, $m_1 < m_2$ if the first variable with different exponents in m_1 and m_2 has lower exponent in m_1.

The Inverse Lexicographical Ordering (InvLex)

Define $m_1 < m_2$ if $e_1^i < e_2^i$ for some $1 \leqslant i \leqslant n$ and $e_1^j = e_2^j$ for all $i < j \leqslant n$. In words, $m_1 < m_2$ if the last variable with different exponents in m_1 and m_2 has lower exponent in m_1.

The Degree Lexicographical Ordering (DegLex)

Define $m_1 < m_2$ if $\deg(m_1) < \deg(m_2)$ or if $\deg(m_1) = \deg(m_2)$ and $m_1 < m_2$ in the Lexicographic Ordering.

Remark 1.2.3 The DegLex ordering is also known as the TLex ordering (T for total degree).

The Degree Inverse Lexicographical Ordering (DegInvLex)

Define $m_1 < m_2$ if $\deg(m_1) < \deg(m_2)$ or if $\deg(m_1) = \deg(m_2)$ and $m_1 < m_2$ in the Inverse Lexicographical Ordering.

The Degree Reverse Lexicographical Ordering (DegRevLex)

Define $m_1 < m_2$ if $\deg(m_1) < \deg(m_2)$ or if $\deg(m_1) = \deg(m_2)$ and $m_1 < m_2$ in the Reverse Lexicographical Ordering, where $m_1 < m_2$ if the last variable with different exponents in m_1 and m_2 has higher exponent in m_1 ($e_1^i > e_2^i$ for some $1 \leqslant i \leqslant n$ and $e_1^j = e_2^j$ for all $i < j \leqslant n$).

Remark 1.2.4 On its own, the Reverse Lexicographical Ordering (RevLex) is not admissible, as $1 > m$ for any monomial $m \neq 1$.

Example 1.2.5 With $x > y > z$, consider the monomials $m_1 := x^2yz$; $m_2 := x^2$ and $m_3 := xyz^2$, with corresponding multidegrees $e_1 = (2, 1, 1)$; $e_2 = (2, 0, 0)$ and $e_3 = (1, 1, 2)$. The following table shows the order placed on the monomials by the various monomial orderings defined above. The final column shows the order induced on the polynomial $p := m_1 + m_2 + m_3$ by the chosen monomial ordering.

Monomial Ordering O	O(m_1, m_2)	O(m_1, m_3)	O(m_2, m_3)	p
Lex	0	0	0	$x^2yz + x^2 + xyz^2$
InvLex	0	1	1	$xyz^2 + x^2yz + x^2$
DegLex	0	0	1	$x^2yz + xyz^2 + x^2$
DegInvLex	0	1	1	$xyz^2 + x^2yz + x^2$
DegRevLex	0	0	1	$x^2yz + xyz^2 + x^2$

1.2.2 Noncommutative Monomial Orderings

In the noncommutative case, because we use words and not multidegrees to represent monomials, our definitions for the lexicographically based orderings will have to be adapted slightly. All other definitions and conventions will stay the same.

The Lexicographic Ordering (Lex)

Define $m_1 < m_2$ if, working left-to-right, the first (say i-th) letter on which m_1 and m_2 differ is such that the i-th letter of m_1 is lexicographically *less* than the i-th letter of m_2 in the variable ordering. Note: this ordering is *not* admissible (counterexample: if $x > y$ is the variable ordering, then $x < xy$ but $x^2 > xyx$).

Remark 1.2.6 When comparing two monomials m_1 and m_2 such that m_1 is a proper prefix of m_2 (for example $m_1 := x$ and $m_2 := xy$ as in the above counterexample), a problem arises with the above definition in that we eventually run out of letters in the shorter word to compare with (in the example, having seen that the first letter of both monomials match, what do we compare the second letter of m_2 with?). One answer is to introduce a padding symbol \$ to pad m_1 on the right to make sure it is the same length as m_2, with the convention that any letter is greater than the padding symbol (so that $m_1 < m_2$). The padding symbol will not explicitly appear anywhere in the remainder of this thesis, but we will bear in mind that it can be introduced to deal with situations where prefixes and suffixes of monomials are involved.

Remark 1.2.7 The lexicographic ordering is also known as the dictionary ordering since the words in a dictionary (such as the Oxford English Dictionary) are ordered using the lexicographic ordering with variable (or alphabetical) ordering $a < b < c < \cdots$. Note however that while a dictionary orders words in increasing order, we will write polynomials in decreasing order.

The Inverse Lexicographical Ordering (InvLex**)**

Define $m_1 < m_2$ if, working left-to-right, the first (say i-th) letter on which m_1 and m_2 differ is such that the i-th letter of m_1 is lexicographically *greater* than the i-th letter of m_2. Note: this ordering (like Lex) is *not* admissible (counterexample: if $x > y$ is the variable ordering, then $xy < x$ but $xyx > x^2$).

The Degree Reverse Lexicographical Ordering (DegRevLex**)**

Define $m_1 < m_2$ if $\deg(m_1) < \deg(m_2)$ or if $\deg(m_1) = \deg(m_2)$ and $m_1 < m_2$ in the Reverse Lexicographical Ordering, where $m_1 < m_2$ if, working in *reverse*, or from right-to-left, the first (say i-th) letter on which m_1 and m_2 differ is such that the i-th letter of m_1 is lexicographically *greater* than the i-th letter of m_2.

Example 1.2.8 With $x > y > z$, consider the noncommutative monomials $m_1 := zxyx$; $m_2 := xzx$ and $m_3 := y^2zx$. The following table shows the order placed on the monomials by various noncommutative monomial orderings. As before, the final column shows the order induced on the polynomial $p := m_1 + m_2 + m_3$ by the chosen monomial ordering.

Monomial Ordering O	O(m_1, m_2)	O(m_1, m_3)	O(m_2, m_3)	p
Lex	1	1	0	$xzx + y^2zx + zxyx$
InvLex	0	0	1	$zxyx + y^2zx + xzx$
DegLex	0	1	1	$y^2zx + zxyx + xzx$
DegInvLex	0	0	1	$zxyx + y^2zx + xzx$
DegRevLex	0	1	1	$y^2zx + zxyx + xzx$

1.2.3 Polynomial Division

Definition 1.2.9 Let $\mathcal{R}$ be a polynomial ring, and let O be an arbitrary admissible monomial ordering. Given two nonzero polynomials p_1, $p_2 \in \mathcal{R}$, we say that p_1 divides

p_2 (written $p_1 \mid p_2$) if the lead monomial of p_1 divides some monomial m (with coefficient c) in p_2. For a commutative polynomial ring, this means that $m = \mathrm{LM}(p_1)m'$ for some monomial m'; for a noncommutative polynomial ring, this means that $m = m_\ell \mathrm{LM}(p_1) m_r$ for some monomials m_ℓ and m_r ($\mathrm{LM}(p_1)$ is a subword of m).

To perform the division, we take away an appropriate multiple of p_1 from p_2 in order to cancel off $\mathrm{LT}(p_1)$ with the term involving m in p_2. In the commutative case, we do

$$p_2 - (c\mathrm{LC}(p_1)^{-1})p_1 m';$$

in the noncommutative case, we do

$$p_2 - (c\mathrm{LC}(p_1)^{-1})m_\ell p_1 m_r.$$

It is clear that the coefficient rings of our polynomial rings have to be division rings in order for the above expressions to be valid, and so we make the following assumption about the polynomial rings we will encounter in the remainder of this thesis.

Remark 1.2.10 From now on, all coefficient rings of polynomial rings will be fields unless otherwise stated.

Example 1.2.11 Let $p_1 := 5z^2x + 2y^2 + x + 4$ and $p_2 := 3xyxz^2x^3 + 2x^2$ be two DegLex ordered polynomials over the noncommutative polynomial ring $\mathbb{Q}\langle x, y, z\rangle$. Because $\mathrm{LM}(p_2) = xyx(z^2x)x^2$, it is clear that $p_1 \mid p_2$, with the quotient and the remainder of the division being

$$q := \left(\tfrac{3}{5}\right) xyx(5z^2x + 2y^2 + x + 4)x^2$$

and

$$\begin{aligned} r &:= 3xyxz^2x^3 + 2x^2 - \left(\tfrac{3}{5}\right) xyx(5z^2x + 2y^2 + x + 4)x^2 \\ &= 3xyxz^2x^3 + 2x^2 - 3xyxz^2x^3 - \left(\tfrac{6}{5}\right) xyxy^2x^2 - \left(\tfrac{3}{5}\right) xyx^4 - \left(\tfrac{12}{5}\right) xyx^3 \\ &= -\left(\tfrac{6}{5}\right) xyxy^2x^2 - \left(\tfrac{3}{5}\right) xyx^4 - \left(\tfrac{12}{5}\right) xyx^3 + 2x^2 \end{aligned}$$

respectively.

Now that we know how to divide one polynomial by another, what does it mean for a polynomial to be divided by a set of polynomials?

Definition 1.2.12 Let $\mathcal{R}$ be a polynomial ring, and let O be an arbitrary admissible

monomial ordering. Given a nonzero polynomial $p \in \mathcal{R}$ and a set of nonzero polynomials $P = \{p_1, p_2, \ldots, p_m\}$, with $p_i \in \mathcal{R}$ for all $1 \leqslant i \leqslant m$, we divide p by P by working through p term by term, testing to see if each term is divisible by any of the p_i in turn. We recursively divide the remainder of each division using the same method until no more divisions are possible, in which case the remainder is either 0 or is *irreducible*.

Algorithms to divide a polynomial p by a set of polynomials P in the commutative and noncommutative cases are given below as Algorithms 1 and 2 respectively. Note that they take advantage of the fact that if the first N terms of a polynomial q are irreducible with respect to P, then the first N terms of any reduction of q will also be irreducible with respect to P.

Algorithm 1 The Commutative Division Algorithm

Input: A nonzero polynomial p and a set of nonzero polynomials $P = \{p_1, \ldots, p_m\}$ over a polynomial ring $R[x_1, \ldots x_n]$; an admissible monomial ordering O.

Output: $\mathrm{Rem}(p, P) := r$, the remainder of p with respect to P.

```
r = 0;
while (p ≠ 0) do
  u = LM(p); c = LC(p); j = 1; found = false;
  while (j ⩽ m) and (found == false) do
    if (LM(p_j) | u) then
      found = true; u' = u/LM(p_j); p = p − (cLC(p_j)^{−1})p_j u';
    else
      j = j + 1;
    end if
  end while
  if (found == false) then
    r = r + LT(p); p = p − LT(p);
  end if
end while
return r;
```

Remark 1.2.13 All algorithms in this thesis use the conventions that '=' denotes an assignment and '==' denotes a test.

Algorithm 2 The Noncommutative Division Algorithm

To divide a nonzero polynomial p with respect to a set of nonzero polynomials $P = \{p_1, \ldots, p_m\}$, where p and the p_i are elements of a noncommutative polynomial ring $R\langle x_1, \ldots, x_n\rangle$, we apply Algorithm 1 with the following changes.

(a) In the inputs, replace the commutative polynomial ring $R[x_1, \ldots x_n]$ by the noncommutative polynomial ring $R\langle x_1, \ldots, x_n\rangle$.

(b) Change the first **if** condition to read

if $(\mathrm{LM}(p_j) \mid u)$ **then**
  found = true;
  choose u_ℓ and u_r such that $u = u_\ell \mathrm{LM}(p_j) u_r$;
  $p = p - (c\mathrm{LC}(p_j)^{-1}) u_\ell p_j u_r$;
else
  $j = j + 1$;
end if

Remark 1.2.14 In Algorithm 2, if there are several candidates for u_ℓ (and therefore for u_r) in the line 'choose u_ℓ and u_r such that $u = u_\ell \mathrm{LM}(p_j) u_r$', the convention in this thesis will be to choose the u_ℓ with the smallest degree.

Example 1.2.15 To demonstrate that the process of dividing a polynomial by a set of polynomials does not necessarily give a unique result, consider the polynomial $p := xyz+x$ and the set of polynomials $P := \{p_1,\ p_2\} = \{xy - z,\ yz + 2x + z\}$, all polynomials being ordered by DegLex and originating from the polynomial ring $\mathbb{Q}[x, y, z]$. If we choose to divide p by p_1 to begin with, we see that p reduces to $xyz + x - (xy - z)z = z^2 + x$, which is irreducible. But if we choose to divide p by p_2 to begin with, we see that p reduces to $xyz + x - (yz + 2x + z)x = -2x^2 - xz + x$, which is again irreducible. This gives rise to the question of which answer (if any!) is the correct one here? In Chapter 2, we will discover that one way of obtaining a unique answer to this question will be to calculate a *Gröbner Basis* for the dividing set P.

Definition 1.2.16 In order to describe how one polynomial is obtained from another through the process of division, we introduce the following notation.

(a) If the polynomial r is obtained by dividing a polynomial p by a polynomial q, then we will use the notation $p \to r$ or $p \to_q r$ (with the latter notation used if we wish to

show how r is obtained from p).

(b) If the polynomial r is obtained by dividing a polynomial p by a sequence of polynomials $q_1, q_2, \ldots, q_\alpha$, then we will use the notation $p \xrightarrow{*} r$.

(c) If the polynomial r is obtained by dividing a polynomial p by a set of polynomials Q, then we will use the notation $p \to_Q r$.

Chapter 2

Commutative Gröbner Bases

Given a basis F generating an ideal J, the central idea in Gröbner Basis theory is to use F to find a basis G for J with the property that the remainder of the division of any polynomial by G is unique. Such a basis is known as a *Gröbner Basis*.

In particular, if a polynomial p is a member of the ideal J, then the remainder of the division of p by a Gröbner Basis G for J is always zero. This gives us a way to solve the Ideal Membership Problem for J – if the remainder of the division of a polynomial p by G is zero, then $p \in J$ (otherwise $p \notin J$).

2.1 S-polynomials

How do we determine whether or not an arbitrary basis F generating an ideal J is a Gröbner Basis? Using the informal definition shown above, in order to show that a basis is *not* a Gröbner Basis, it is sufficient to find a polynomial p whose remainder on division by F is non-unique. Let us now construct an example in which this is the case, and let us analyse what can to be done to eliminate the non-uniqueness of the remainder.

Let $p_1 = a_1 + a_2 + \cdots + a_\alpha$; $p_2 = b_1 + b_2 + \cdots + b_\beta$ and $p_3 = c_1 + c_2 + \cdots + c_\gamma$ be three polynomials ordered with respect to some fixed admissible monomial ordering O (the a_i, b_j and c_k are all nontrivial terms). Assume that $p_1 \mid p_3$ and $p_2 \mid p_3$, so that we are able to take away from p_3 multiples s and t of p_1 and p_2 respectively to obtain remainders r_1

and r_2.

$$\begin{aligned}
r_1 &= p_3 - sp_1 \\
&= c_1 + c_2 + \cdots + c_\gamma - s(a_1 + a_2 + \cdots + a_\alpha) \\
&= c_2 + \cdots + c_\gamma - sa_2 - \cdots - sa_\alpha; \\
r_2 &= p_3 - tp_2 \\
&= c_2 + \cdots + c_\gamma - tb_2 - \cdots - tb_\beta.
\end{aligned}$$

If we assume that r_1 and r_2 are irreducible and that $r_1 \neq r_2$, it is clear that the remainder of the division of the polynomial p_3 by the set of polynomials $P = \{p_1, p_2\}$ is non-unique, from which we deduce that P is not a Gröbner Basis for the ideal that it generates. We must therefore change P in some way in order for it to become a Gröbner Basis, but what changes are required and indeed allowed?

Consider that we want to add a polynomial to P. To avoid changing the ideal that is being generated by P, any polynomial added to P must be a member of the ideal. It is clear that r_1 and r_2 are members of the ideal, as is the polynomial $p_4 = r_2 - r_1 = -tp_2 + sp_1$. Consider that we add p_4 to P, so that P becomes the set

$$\{a_1 + a_2 + \cdots + a_\alpha,\ b_1 + b_2 + \cdots + b_\beta,\ -tb_2 - tb_3 - \cdots - tb_\beta + sa_2 + sa_3 + \cdots + sa_\alpha\}.$$

If we now divide the polynomial p_3 by the enlarged set P, to begin with (as before) we can either divide p_3 by p_1 or p_2 to obtain remainders r_1 or r_2. Here however, if we assume (without loss of generality[1]) that $\mathrm{LT}(p_4) = -tb_2$, we can now divide r_2 by p_4 to obtain a new remainder

$$\begin{aligned}
r_3 &= r_2 - p_4 \\
&= c_2 + \cdots + c_\gamma - tb_2 - \cdots - tb_\beta - (-tb_2 - tb_3 - \cdots - tb_\beta + sa_2 + sa_3 + \cdots + sa_\alpha) \\
&= c_2 + \cdots + c_\gamma - sa_2 - \cdots - sa_\alpha \\
&= r_1.
\end{aligned}$$

It follows that by adding p_4 to P, we have ensured that the remainder of the division of p_3 by P is unique[2] no matter which of the polynomials p_1 and p_2 we choose to divide

[1] The other possible case is $\mathrm{LT}(p_4) = sa_2$, in which case it is r_1 that reduces to r_2 and not r_2 to r_1.

[2] This may not strictly be true if p_3 is divisible by p_4; for the time being we shall assume that this is not the case, noting that the important concept here is of eliminating the non-uniqueness given by the

p_3 by first. This solves our original problem of non-unique remainders in this restricted situation.

At first glance, the polynomial added to P to solve this problem is dependent upon the polynomial p_3. The reason for saying this is that the polynomial added to P has the form $p_4 = sp_1 - tp_2$, where s and t are terms chosen to multiply the polynomials p_1 and p_2 so that the lead terms of sp_1 and tp_2 equal $\mathrm{LT}(p_3)$ (in fact $s = \frac{\mathrm{LT}(p_3)}{\mathrm{LT}(p_1)}$ and $t = \frac{\mathrm{LT}(p_3)}{\mathrm{LT}(p_2)}$).

However, by definition, $\mathrm{LM}(p_3)$ is a common multiple of $\mathrm{LM}(p_1)$ and $\mathrm{LM}(p_2)$. Because all such common multiples are multiples of the least common multiple of $\mathrm{LM}(p_1)$ and $\mathrm{LM}(p_2)$ (so that $\mathrm{LM}(p_3) = \mu(\mathrm{lcm}(\mathrm{LM}(p_1), \mathrm{LM}(p_2)))$ for some monomial μ), it follows that we can rewrite p_4 as

$$p_4 = \mathrm{LC}(p_3)\mu\left(\frac{\mathrm{lcm}(\mathrm{LM}(p_1), \mathrm{LM}(p_2))}{\mathrm{LT}(p_1)}p_1 - \frac{\mathrm{lcm}(\mathrm{LM}(p_1), \mathrm{LM}(p_2))}{\mathrm{LT}(p_2)}p_2\right).$$

Consider now that we add the polynomial $p_5 = \frac{p_4}{\mathrm{LC}(p_3)\mu}$ to P instead of adding p_4 to P. It follows that even though this polynomial does not depend on the polynomial p_3, we can still obtain a unique remainder when dividing p_3 by p_1 and p_2, because we can do $r_3 = r_2 - \mathrm{LC}(p_3)\mu p_5$. Moreover, the polynomial p_5 solves the problem of non-unique remainders for *any* polynomial p_3 that is divisible by both p_1 and p_2 (all that changes is the multiple of p_5 used in the reduction of r_2); we call such a polynomial an *S-polynomial*[3] for p_1 and p_2.

Definition 2.1.1 The *S-polynomial* of two distinct polynomials p_1 and p_2 is given by the expression

$$\text{S-pol}(p_1, p_2) = \frac{\mathrm{lcm}(\mathrm{LM}(p_1), \mathrm{LM}(p_2))}{\mathrm{LT}(p_1)}p_1 - \frac{\mathrm{lcm}(\mathrm{LM}(p_1), \mathrm{LM}(p_2))}{\mathrm{LT}(p_2)}p_2.$$

Remark 2.1.2 The terms $\frac{\mathrm{lcm}(\mathrm{LM}(p_1),\mathrm{LM}(p_2))}{\mathrm{LT}(p_1)}$ and $\frac{\mathrm{lcm}(\mathrm{LM}(p_1),\mathrm{LM}(p_2))}{\mathrm{LT}(p_2)}$ can be thought of as the terms used to multiply the polynomials p_1 and p_2 so that the lead monomials of the multiples are equal to the monomial $\mathrm{lcm}(\mathrm{LM}(p_1), \mathrm{LM}(p_2))$.

Let us now illustrate how adding an S-polynomial to a basis solves the problem of non-unique remainders in a particular example.

choice of dividing p_3 by p_1 or p_2 first.

[3] The S stands for Syzygy, as in a pair of connected objects.

Example 2.1.3 Recall that in Example 1.2.15 we showed how dividing the polynomial $p := xyz + x$ by the two polynomials in the set $P := \{p_1, p_2\} = \{xy - z,\ yz + 2x + z\}$ gave two different remainders, $r_1 := z^2 + x$ and $r_2 := -2x^2 - xz + x$ respectively. Consider now that we add S-pol(p_1, p_2) to P, where

$$\begin{aligned} \text{S-pol}(p_1, p_2) &= \frac{xyz}{xy}(xy - z) - \frac{xyz}{yz}(yz + 2x + z) \\ &= (xyz - z^2) - (xyz + 2x^2 + xz) \\ &= -2x^2 - xz - z^2. \end{aligned}$$

Dividing p by the enlarged set, if we choose to divide p by p_1 to begin with, we see that p reduces (as before) to give $xyz + x - (xy - z)z = z^2 + x$, which is irreducible. Similarly, dividing p by p_2 to begin with, we obtain the remainder $xyz + x - (yz + 2x + z)x = -2x^2 - xz + x$. However, whereas before this remainder was irreducible, now we can reduce it by the S-polynomial to give $-2x^2 - xz + x - (-2x^2 - xz - z^2) = z^2 + x$, which is equal to the first remainder.

Let us now formally define a Gröbner Basis in terms of S-polynomials, noting that there are many other equivalent definitions (see for example [7], page 206).

Definition 2.1.4 Let $G = \{g_1, \ldots, g_m\}$ be a basis for an ideal J over a commutative polynomial ring $\mathcal{R} = R[x_1, \ldots, x_n]$. If all the S-polynomials involving members of G reduce to zero using G (S-pol$(g_i, g_j) \rightarrow_G 0$ for all $i \neq j$), then G is a *Gröbner Basis* for J.

Theorem 2.1.5 *Given any polynomial p over a polynomial ring $\mathcal{R} = R[x_1, \ldots, x_n]$, the remainder of the division of p by a basis G for an ideal J in $\mathcal{R}$ is unique if and only if G is a Gröbner Basis.*

Proof: ($\Rightarrow$) By Newman's Lemma (cf. [7], page 176), showing that the remainder of the division of p by G is unique is equivalent to showing that the division process is *locally confluent*, that is if there are polynomials $f,\ f_1,\ f_2 \in \mathcal{R}$ with $f_1 = f - t_1 g_1$ and $f_2 = f - t_2 g_2$ for terms t_1, t_2 and $g_1, g_2 \in G$, then there exists a polynomial $f_3 \in \mathcal{R}$ such that both f_1 and f_2 reduce to f_3. By the Translation Lemma (cf. [7], page 200), this in turn is equivalent to showing that the polynomial $f_2 - f_1 = t_1 g_1 - t_2 g_2$ reduces to zero, which is what we shall now do.

There are two cases to deal with, $\mathrm{LT}(t_1g_1) \neq \mathrm{LT}(t_2g_2)$ and $\mathrm{LT}(t_1g_1) = \mathrm{LT}(t_2g_2)$. In the first case, notice that the remainders f_1 and f_2 are obtained by cancelling off different terms of the original f (the reductions of f are *disjoint*), so it is possible, assuming (without loss of generality) that $\mathrm{LT}(t_1g_1) > \mathrm{LT}(t_2g_2)$, to directly reduce the polynomial $f_2 - f_1 = t_1g_1 - t_2g_2$ in the following manner: $t_1g_1 - t_2g_2 \rightarrow_{g_1} -t_2g_2 \rightarrow_{g_2} 0$. In the second case, the reductions of f are not disjoint (as the same term t from f is cancelled off during both reductions), so that the term t does not appear in the polynomial $t_1g_1 - t_2g_2$. However, the term t is a common multiple of $\mathrm{LT}(t_1g_1)$ and $\mathrm{LT}(t_2g_2)$, and thus the polynomial $t_1g_1 - t_2g_2$ is a multiple of the S-polynomial $\text{S-pol}(g_1, g_2)$, say

$$t_1g_1 - t_2g_2 = \mu(\text{S-pol}(g_1, g_2))$$

for some term μ. Because G is a Gröbner Basis, the S-polynomial $\text{S-pol}(g_1, g_2)$ reduces to zero, and hence by extension the polynomial $t_1g_1 - t_2g_2$ also reduces to zero.

($\Leftarrow$) As all S-polynomials are members of the ideal J, to complete the proof it is sufficient to show that there is always a reduction path of an arbitrary member of the ideal that leads to a zero remainder (the uniqueness of remainders will then imply that members of the ideal will always reduce to zero). Let $f \in J = \langle G \rangle$. Then, by definition, there exist $g_i \in G$ and $f_i \in \mathcal{R}$ (where $1 \leqslant i \leqslant j$) such that

$$f = \sum_{i=1}^{j} f_i g_i.$$

We proceed by induction on j. If $j = 1$, then $f = f_1g_1$, and it is clear that we can use g_1 to reduce f to give a zero remainder ($f \rightarrow f - f_1g_1 = 0$). Assume that the result is true for $j = k$, and let us look at the case $j = k + 1$, so that

$$f = \left(\sum_{i=1}^{k} f_i g_i \right) + f_{k+1}g_{k+1}.$$

By the inductive hypothesis, $\sum_{i=1}^{k} f_i g_i$ is a member of the ideal that reduces to zero. The polynomial f therefore reduces to the polynomial $f' := f_{k+1}g_{k+1}$, and we can now use g_{k+1} to reduce f' to give a zero remainder ($f' \rightarrow f' - f_{k+1}g_{k+1} = 0$). □

We are now in a position to be able to define an algorithm to compute a Gröbner Basis. However, to be able to prove that this algorithm always terminates, we must first prove

a result stating that all ideals over commutative polynomial rings are finitely generated. This proof takes place in two stages – first for monomial ideals (Dickson's Lemma) and then for polynomial ideals (Hilbert's Basis Theorem).

2.2 Dickson's Lemma and Hilbert's Basis Theorem

Definition 2.2.1 A *monomial ideal* is an ideal generated by a set of monomials.

Remark 2.2.2 Any polynomial p that is a member of a monomial ideal is a sum of terms $p = \sum_i t_i$, where each t_i is a member of the monomial ideal.

Lemma 2.2.3 (Dickson's Lemma) *Every monomial ideal over the polynomial ring* $\mathcal{R} = R[x_1, \ldots, x_n]$ *is finitely generated.*

Proof (cf. [22], page 47): Let J be a monomial ideal over $\mathcal{R}$ generated by a set of monomials S. We proceed by induction on n, our goal being to show that S always has a finite subset T generating J. For $n = 1$, notice that all elements of S will be of the form x_1^j for some $j \geqslant 0$. Let T be the singleton set containing the member of S with the lowest degree (that is the x_1^j with the lowest value of j). Clearly T is finite, and because any element of S is a multiple of the chosen x_1^j, it is also clear that T generates the same ideal as S.

For the inductive step, assume that all monomial ideals over the polynomial ring $\mathcal{R}' = R[x_1, \ldots, x_{n-1}]$ are finitely generated. Let $C_0 \subseteq C_1 \subseteq C_2 \subseteq \cdots$ be an ascending chain of monomial ideals over $\mathcal{R}'$, where[4]

$$C_j = \langle S_j \rangle \cap \mathcal{R}',\ S_j = \left\{ \frac{s}{\gcd(s, x_n^j)} \mid s \in S \right\}.$$

Let the monomial m be an arbitrary member of the ideal J, expressed as $m = m' x_n^k$, where $m' \in \mathcal{R}'$ and $k \geqslant 0$. By definition, $m' \in C_k$, and so $m \in x_n^k C_k$. By the inductive hypothesis, each C_k is finitely generated by a set T_k, and so $m \in x_n^k \langle T_k \rangle$. From this we can deduce that

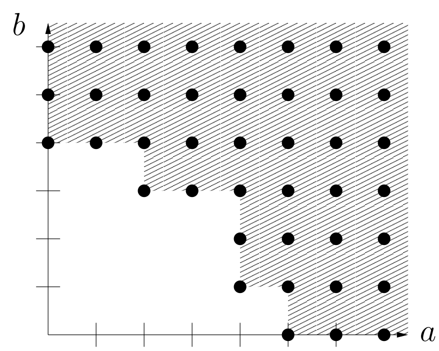
$$T = T_0 \cup x_n T_1 \cup x_n^2 T_2 \cup \cdots$$

is a generating set for J.

[4]Think of C_0 as the set of monomials $m \in J$ which are also members of $\mathcal{R}'$; think of C_j (for $j \geqslant 1$) as containing all the elements of C_{j-1} plus the monomials $m \in J$ of the form $m = m' x_n^j$, $m' \in \mathcal{R}'$.

Consider the ideal $C = \cup C_j$ for $j \geqslant 0$. This is another monomial ideal over $\mathcal{R}'$, and so by the inductive hypothesis is finitely generated. It follows that the chain must stop as soon as the generators of C are contained in some C_r, so that $C_r = C_{r+1} = \cdots$ (and hence $T_r = T_{r+1} = \cdots$). It follows that $T_0 \cup x_n T_1 \cup x_n^2 T_2 \cup \cdots \cup x_n^r T_r$ is a finite subset of S generating J. □

Example 2.2.4 Let $S = \{y^4,\ xy^4,\ x^2y^3,\ x^3y^3,\ x^4y,\ x^k\}$ be an infinite set of monomials generating an ideal J over the polynomial ring $\mathbb{Q}[x, y]$, where k is an integer such that $k \geqslant 5$. We can visualise J by using the following monomial lattice, where a point (a, b) in the lattice (for non-negative integers a, b) corresponds to the monomial $x^a y^b$, and the shaded region contains all monomials which are reducible by some member of S (and hence belong to J).



To show that J can be finitely generated, we need to construct the set T as described in the proof of Dickson's Lemma. The first step in doing this is to construct the sequence of sets $S_j = \left\{\frac{s}{\gcd(s, y^j)} \mid s \in S\right\}$ for all $j \geqslant 0$.

$$\begin{array}{rcl}
S_0 & = & \{y^4,\ xy^4,\ x^2y^3,\ x^3y^3,\ x^4y,\ x^k\} = S \\
S_1 & = & \{y^3,\ xy^3,\ x^2y^2,\ x^3y^2,\ x^4,\ x^k\} \\
S_2 & = & \{y^2,\ xy^2,\ x^2y,\ x^3y,\ x^4,\ x^k\} \\
S_3 & = & \{y,\ xy,\ x^2,\ x^3,\ x^4,\ x^k\} \\
S_4 & = & \{y^0 = 1,\ x,\ x^2,\ x^3,\ x^4,\ x^k\} \\
S_{j+1} & = & S_j \ \text{ for all } j+1 \geqslant 5.
\end{array}$$

Each set S_j gives rise to an ideal C_j consisting of all monomials $m \in \langle S_j \rangle$ of the form $m = x^i$ for some $i \geqslant 0$. Because each of these ideals is an ideal over the polynomial ring $\mathbb{Q}[x]$, we can use an inductive hypothesis to give us a finite generating set T_j for each C_j.

In this case, the first paragraph of the proof of Dickson's Lemma tells us how to apply the inductive hypothesis — each set T_j is formed by choosing the monomial $m \in S_j$ of lowest degree such that $m = x^i$ for some $i \geqslant 0$.

$$\begin{aligned} T_0 &= \{x^5\} \\ T_1 &= \{x^4\} \\ T_2 &= \{x^4\} \\ T_3 &= \{x^2\} \\ T_4 &= \{x^0 = 1\} \\ T_{j+1} &= T_j \text{ for all } k+1 \geqslant 5. \end{aligned}$$

We can now deduce that

$$T = \{x^5\} \cup \{x^4y\} \cup \{x^4y^2\} \cup \{x^2y^3\} \cup \{y^4\} \cup \{y^5\} \cup \cdots$$

is a generating set for J. Further, because $T_{j+1} = T_j$ for all $k+1 \geqslant 5$, we can also deduce that the set

$$T' = \{x^5,\ x^4y,\ x^4y^2,\ x^2y^3,\ y^4\}$$

is a finite generating set for J (a fact that can be verified by drawing a monomial lattice for T' and comparing it with the above monomial lattice for the set S).

Theorem 2.2.5 (Hilbert's Basis Theorem) *Every ideal J over a polynomial ring $\mathcal{R} = R[x_1, \ldots, x_n]$ is finitely generated.*

Proof: Let O be a fixed arbitrary admissible monomial ordering, and define $\mathrm{LM}(J) = \langle \mathrm{LM}(p) \mid p \in J \rangle$. Because $\mathrm{LM}(J)$ is a monomial ideal, by Dickson's Lemma it is finitely generated, say by the set of monomials $M = \{m_1, \ldots, m_r\}$. By definition, each $m_i \in M$ (for $1 \leqslant i \leqslant r$) has a corresponding $p_i \in J$ such that $\mathrm{LM}(p_i) = m_i$. We claim that $P = \{p_1, \ldots, p_r\}$ is a generating set for J. To prove the claim, notice that $\langle P \rangle \subseteq J$ so that $f \in \langle P \rangle \Rightarrow f \in J$. Conversely, given a polynomial $f \in J$, we know that $\mathrm{LM}(f) \in \langle M \rangle$ so that $\mathrm{LM}(f) = \alpha m_j$ for some monomial α and some $1 \leqslant j \leqslant r$. From this, if we define $\alpha' = \frac{\mathrm{LC}(f)}{\mathrm{LC}(p_j)}\alpha$, we can deduce that $\mathrm{LM}(f - \alpha' p_j) < \mathrm{LM}(f)$. Since $f - \alpha' p_j \in J$, and because of the admissibility of O, by recursion on $f - \alpha' p_j$ (define $f_{k+1} = f_k - \alpha'_k p_{j_k}$ for $k \geqslant 1$, where $f_1 - \alpha'_1 p_{j_1} := f - \alpha' p_j$), we can deduce that $f \in \langle P \rangle$ (in fact $f = \sum_{k=1}^{K} \alpha'_k p_{j_k}$ for some finite K). $\square$

Corollary 2.2.6 (The Ascending Chain Condition) *Every ascending sequence of ideals $J_1 \subseteq J_2 \subseteq \cdots$ over a polynomial ring $\mathcal{R} = R[x_1, \ldots, x_n]$ is eventually constant, so that there is an i such that $J_i = J_{i+1} = \cdots$.*

Proof: By Hilbert's Basis Theorem, each ideal J_k (for $k \geqslant 1$) is finitely generated. Consider the ideal $J = \cup J_k$. This is another ideal over $\mathcal{R}$, and so by Hilbert's Basis Theorem is also finitely generated. From this we deduce that the chain must stop as soon as the generators of J are contained in some J_i, so that $J_i = J_{i+1} = \cdots$. □

2.3 Buchberger's Algorithm

The algorithm used to compute a Gröbner Basis is known as Buchberger's Algorithm. Bruno Buchberger was a student of Wolfgang Gröbner at the University of Innsbruck, Austria, and the publication of his PhD thesis in 1965 [11] marked the start of Gröbner Basis theory.

In Buchberger's algorithm, S-polynomials for pairs of elements from the current basis are computed and reduced using the current basis. If the S-polynomial does not reduce to zero, it is added to the current basis, and this process continues until all S-polynomials reduce to zero. The algorithm works on the principle that if an S-polynomial S-pol(g_i, g_j) does not reduce to zero using a set of polynomials G, then it will certainly reduce to zero using the set of polynomials $G \cup \{\text{S-pol}(g_i, g_j)\}$.

Theorem 2.3.1 *Algorithm 3 always terminates with a Gröbner Basis for the ideal J.*

Proof (cf. [7], page 213): *Correctness.* If the algorithm terminates, it does so with a set of polynomials G with the property that all S-polynomials involving members of G reduce to zero using G (S-pol$(g_i, g_j) \to_G 0$ for all $i \neq j$). G is therefore a Gröbner Basis by Definition 2.1.4. *Termination.* If the algorithm does not terminate, then an endless sequence of polynomials must be added to the set G so that the set A never becomes empty. Let $G_0 \subset G_1 \subset G_2 \subset \cdots$ be the successive values of G. If we consider the corresponding sequence $\text{LM}(G_0) \subset \text{LM}(G_1) \subset \text{LM}(G_2) \subset \cdots$ of lead monomials, we note that these sets generate an ascending chain of ideals $J_0 \subset J_1 \subset J_2 \subset \cdots$ because each time we add a monomial to a particular set $\text{LM}(G_k)$ to form the set $\text{LM}(G_{k+1})$, the monomial we choose is irreducible with respect to $\text{LM}(G_k)$, and hence does not belong to the ideal J_k. However the Ascending Chain Condition tells us that such a chain of ideals

Algorithm 3 A Basic Commutative Gröbner Basis Algorithm

Input: A Basis $F = \{f_1, f_2, \ldots, f_m\}$ for an ideal J over a commutative polynomial ring $R[x_1, \ldots x_n]$; an admissible monomial ordering O.

Output: A Gröbner Basis $G = \{g_1, g_2, \ldots, g_p\}$ for J.

Let $G = F$ and let $A = \emptyset$;
For each pair of polynomials (g_i, g_j) in G $(i < j)$,
add the S-polynomial S-pol(g_i, g_j) to A;
while (A is not empty) **do**
  Remove the first entry s_1 from A;
  $s_1' = \text{Rem}(s_1, G)$;
  if $(s_1' \neq 0)$ **then**
    Add s_1' to G and add all the S-polynomials S-pol(g_i, s_1') to A $(g_i \in G,\ g_i \neq s_1')$;
  end if
end while
return G;

must eventually become constant, so there must be some $i \geqslant 0$ such that $J_i = J_{i+1} = \cdots$. It follows that the algorithm will terminate once the set G_i has been constructed, as all of the S-polynomials left in A will now reduce to zero (if not, some S-polynomial left in A will reduce to a non-zero polynomial s_1' whose lead monomial is irreducible with respect to $\text{LM}(G_i)$, allowing us to construct an ideal $J_{i+1} = \langle \text{LM}(G_i) \cup \{\text{LM}(s_1')\} \rangle \supset \langle \text{LM}(G_i) \rangle = J_i$, contradicting the fact that $J_{i+1} = J_i$.) □

Example 2.3.2 Let $F := \{f_1, f_2\} = \{x^2 - 2xy + 3,\ 2xy + y^2 + 5\}$ generate an ideal over the commutative polynomial ring $\mathbb{Q}[x, y]$, and let the monomial ordering be DegLex. Running Algorithm 3 on F, there is only one S-polynomial to consider initially, namely S-pol$(f_1, f_2) = y(f_1) - \frac{1}{2}x(f_2) = -\frac{5}{2}xy^2 - \frac{5}{2}x + 3y$. This polynomial reduces (using f_2) to give the irreducible polynomial $\frac{5}{4}y^3 - \frac{5}{2}x + \frac{37}{4}y =: f_3$, which we add to our current basis. This produces two more S-polynomials to look at, S-pol$(f_1, f_3) = y^3(f_1) - \frac{4}{5}x^2(f_3) = -2xy^4 + 2x^3 - \frac{37}{5}x^2y + 3y^3$ and S-pol$(f_2, f_3) = \frac{1}{2}y^2(f_2) - \frac{4}{5}x(f_3) = \frac{1}{2}y^4 + 2x^2 - \frac{37}{5}xy + \frac{5}{2}y^2$, both of which reduce to zero. The algorithm therefore terminates with the set $\{x^2 - 2xy + 3,\ 2xy + y^2 + 5,\ \frac{5}{4}y^3 - \frac{5}{2}x + \frac{37}{4}y\}$ as the output Gröbner Basis.

Here is a dry run for Algorithm 3 in this instance.

G	i	j	A	s_1	s'_1
$\{f_1, f_2\}$	1	2	$\emptyset$		
			$\{\text{S-pol}(f_1, f_2)\}$		
$\{f_1, f_2, f_3\}$	1		$\emptyset$	$-\frac{5}{2}xy^2 - \frac{5}{2}x + 3y$	f_3
	2		$\{\text{S-pol}(f_1, f_3)\}$		
			$\{\text{S-pol}(f_2, f_3),\ \text{S-pol}(f_1, f_3)\}$		
			$\{\text{S-pol}(f_1, f_3)\}$	$\frac{1}{2}y^4 + 2x^2 - \frac{37}{5}xy + \frac{5}{2}y^2$	0
			$\emptyset$	$-2xy^4 + 2x^3 - \frac{37}{5}x^2y + 3y^3$	0

2.4 Reduced Gröbner Bases

Definition 2.4.1 Let $G = \{g_1, \ldots, g_p\}$ be a Gröbner Basis for an ideal over the polynomial ring $R[x_1, \ldots, x_n]$. G is a *reduced* Gröbner Basis if the following conditions are satisfied.

(a) $\text{LC}(g_i) = 1_R$ for all $g_i \in G$.

(b) No term in any polynomial $g_i \in G$ is divisible by any $\text{LT}(g_j)$, $j \neq i$.

Theorem 2.4.2 *Every ideal over a commutative polynomial ring has a unique reduced Gröbner Basis.*

Proof: *Existence.* By Theorem 2.3.1, there exists a Gröbner Basis G for every ideal over a commutative polynomial ring. We claim that the following procedure transforms G into a reduced Gröbner Basis G'.

(i) Multiply each $g_i \in G$ by $\text{LC}(g_i)^{-1}$.

(ii) Reduce each $g_i \in G$ by $G \setminus \{g_i\}$, removing from G all polynomials that reduce to zero.

It is clear that G' satisfies the conditions of Definition 2.4.1, so it remains to show that G' is a Gröbner Basis, which we shall do by showing that the application of each step of instruction (ii) above produces a basis which is still a Gröbner Basis.

Let $G = \{g_1, \ldots, g_p\}$ be a Gröbner Basis, and let g'_i be the reduction of an arbitrary

$g_i \in G$ with respect to $G \setminus \{g_i\}$, carried out as follows (the t_k are terms).

$$g_i' = g_i - \sum_{k=1}^{\kappa} t_k g_{j_k}. \tag{2.1}$$

Set $H = (G\setminus\{g_i\})\cup\{g_i'\}$ if $g_i' \neq 0$, and set $H = G\setminus\{g_i\}$ if $g_i' = 0$. As G is a Gröbner Basis, all S-polynomials involving elements of G reduce to zero using G, so there are expressions

$$t_a g_a - t_b g_b - \sum_{u=1}^{\mu} t_u g_{c_u} = 0 \tag{2.2}$$

for every S-polynomial S-pol$(g_a, g_b) = t_a g_a - t_b g_b$, where $g_a, g_b, g_{c_u} \in G$. To show that H is a Gröbner Basis, we must show that all S-polynomials involving elements of H reduce to zero using H. For distinct polynomials $g_a, g_b \in H$ not equal to g_i', we can reduce the S-polynomial S-pol(g_a, g_b) using the reduction shown in Equation (2.2), substituting for g_i from Equation (2.1) if any of the g_{c_u} in Equation (2.2) are equal to g_i. This gives a reduction to zero of S-pol(g_a, g_b) in terms of elements of H.

If $g_i' = 0$, our proof is complete. Otherwise consider the S-polynomial S-pol(g_i', g_a). We claim that S-pol$(g_i, g_a) = t_1 g_i - t_2 g_a \Rightarrow$ S-pol$(g_i', g_a) = t_1 g_i' - t_2 g_a$. To prove this claim, it is sufficient to show that $\mathrm{LT}(g_i) = \mathrm{LT}(g_i')$. Assume for a contradiction that $\mathrm{LT}(g_i) \neq \mathrm{LT}(g_i')$. It follows that during the reduction of g_i we were able to reduce its lead term, so that $\mathrm{LT}(g_i) = t\mathrm{LT}(g_j)$ for some term t and some $g_j \in G$. By the admissibility of the chosen monomial ordering, the polynomial $g_i - t g_j$ reduces to zero without using g_i, leading to the conclusion that $g_i' = 0$, a contradiction.

It remains to show that S-pol$(g_i', g_a) \to_H 0$. We know that S-pol$(g_i, g_a) = t_1 g_i - t_2 g_a \to_G 0$, and Equation (2.2) tells us that $t_1 g_i - t_2 g_a - \sum_{u=1}^{\mu} t_u g_{c_u} = 0$. Substituting for g_i from Equation (2.1), we obtain[5]

$$t_1 \left(g_i' + \sum_{k=1}^{\kappa} t_k g_{j_k} \right) - t_2 g_a - \sum_{u=1}^{\mu} t_u g_{c_u} = 0$$

or

$$t_1 g_i' - t_2 g_a - \left(\sum_{u=1}^{\mu} t_u g_{c_u} - \sum_{k=1}^{\kappa} t_1 t_k g_{j_k} \right) = 0,$$

[5] Substitutions for g_i may also occur in the summation $\sum_{u=1}^{\mu} t_u g_{c_u}$; these substitutions have not been considered in the displayed formulae.

which implies that S-pol$(g_i', g_a) \to_H 0$.

Uniqueness. Assume for a contradiction that $G = \{g_1, \ldots, g_p\}$ and $H = \{h_1, \ldots, h_q\}$ are two reduced Gröbner Bases for an ideal J, with $G \neq H$. Let g_i be an arbitrary element from G (where $1 \leqslant i \leqslant p$). Because g_i is a member of the ideal, then g_i must reduce to zero using H (H is a Gröbner Basis). This means that there must exist a polynomial $h_j \in H$ such that $\mathrm{LT}(h_j) \mid \mathrm{LT}(g_i)$. If $\mathrm{LT}(h_j) \neq \mathrm{LT}(g_i)$, then $\mathrm{LT}(h_j) \times m = \mathrm{LT}(g_i)$ for some nontrivial monomial m. But h_j is also a member of the ideal, so it must reduce to zero using G. Therefore there exists a polynomial $g_k \in G$ such that $\mathrm{LT}(g_k) \mid \mathrm{LT}(h_j)$, which implies that $\mathrm{LT}(g_k) \mid \mathrm{LT}(g_i)$, with $k \neq i$. This contradicts condition (b) of Definition 2.4.1, so that G cannot be a reduced Gröbner Basis for J if $\mathrm{LT}(h_j) \neq \mathrm{LT}(g_i)$. From this we deduce that each $g_i \in G$ has a corresponding $h_j \in H$ such that $\mathrm{LT}(g_i) = \mathrm{LT}(h_j)$. Further, because G and H are assumed to be reduced Gröbner Bases, this is a one-to-one correspondence.

It remains to show that if $\mathrm{LT}(g_i) = \mathrm{LT}(h_j)$, then $g_i = h_j$. Assume for a contradiction that $g_i \neq h_j$, and consider the polynomial $g_i - h_j$. Without loss of generality, assume that $\mathrm{LM}(g_i - h_j)$ appears in g_i. Because $g_i - h_j$ is a member of the ideal, then there is a polynomial $g_k \in G$ such that $\mathrm{LT}(g_k) \mid \mathrm{LT}(g_i - h_j)$. But this again contradicts condition (b) of Definition 2.4.1, as we have shown that there is a term in g_i that is divisible by $\mathrm{LT}(g_k)$ for some $k \neq i$. It follows that G cannot be a reduced Gröbner Basis if $g_i \neq h_j$, which means that $G = H$ and therefore reduced Gröbner Bases are unique. □

Given a Gröbner Basis G, we saw in the proof of Theorem 2.4.2 that if the lead term of any polynomial $g_i \in G$ is reducible by some polynomial $g_j \in G$ (where $g_j \neq g_i$), then g_i reduces to zero. We can use this information to refine the procedure for finding a unique reduced Gröbner Basis (as given in the aforementioned proof) by allowing the removal of any polynomial $g_i \in G$ whose lead monomial is a multiple of some other lead monomial $\mathrm{LM}(g_j)$. This process, which if often referred to as *minimising* a Gröbner Basis (as in finding a Gröbner Basis with the minimal number of elements), is incorporated into our refined procedure, which we state as Algorithm 4.

2.5 Improvements to Buchberger's Algorithm

Nowadays, most general purpose symbolic computation systems possess an implementation of Buchberger's algorithm. These implementations often take advantage of the

Algorithm 4 The Commutative Unique Reduced Gröbner Basis Algorithm

Input: A Gröbner Basis $G = \{g_1, g_2, \ldots, g_m\}$ for an ideal J over a commutative polynomial ring $R[x_1, \ldots x_n]$; an admissible monomial ordering O.

Output: The unique reduced Gröbner Basis $G' = \{g'_1, g'_2, \ldots, g'_p\}$ for J.

$G' = \emptyset$;
for each $g_i \in G$ **do**
  Multiply g_i by $\mathrm{LC}(g_i)^{-1}$;
  if $(\mathrm{LM}(g_i) = u\mathrm{LM}(g_j)$ for some monomial u and some $g_j \in G$ $(g_j \neq g_i))$ **then**
    $G = G \setminus \{g_i\}$;
  end if
end for
for each $g_i \in G$ **do**
  $g'_i = \mathrm{Rem}(g_i, (G \setminus \{g_i\}) \cup G')$;
  $G = G \setminus \{g_i\}$; $G' = G' \cup \{g'_i\}$;
end for
return G';

numerous improvements made to Buchberger's algorithm over the years, some of which we shall now describe.

2.5.1 Buchberger's Criteria

In 1979, Buchberger published a paper [10] which gave criteria that enable the *a priori* detection of S-polynomials that reduce to zero. This speeds up Algorithm 3 by drastically reducing the number of S-polynomials that must be reduced with respect to the current basis.

Proposition 2.5.1 (Buchberger's First Criterion) *Let f and g be two polynomials over a commutative polynomial ring ordered with respect to some fixed admissible monomial ordering O. If the lead terms of f and g are disjoint (so that* $\mathrm{lcm}(\mathrm{LM}(f), \mathrm{LM}(g)) = \mathrm{LM}(f)\mathrm{LM}(g)$*), then* $\text{S-pol}(f, g)$ *reduces to zero using the set* $\{f, g\}$.

Proof (Adapted from [7], Lemma 5.66): Assume that $f = \sum_{i=1}^{\alpha} s_i$ and $g = \sum_{j=1}^{\beta} t_j$, where the s_i and the t_j are terms. Because s_1 and t_1 are disjoint, it follows that

$$\begin{aligned} \text{S-pol}(f,g) &\equiv t_1 f - s_1 g \\ &= t_1(s_2 + \cdots + s_\alpha) - s_1(t_2 + \cdots + t_\beta). \end{aligned} \tag{2.3}$$

We claim that no two terms in Equation (2.3) are the same. Assume to the contrary that $t_1 s_i = s_1 t_j$ for some $2 \leqslant i \leqslant \alpha$ and $2 \leqslant j \leqslant \beta$. Then $t_1 s_i$ is a multiple of both t_1 and s_1, which means that $t_1 s_i$ is a multiple of $\text{lcm}(t_1, s_1) = t_1 s_1$. But then we must have $t_1 s_i \geqslant t_1 s_1$, which gives a contradiction (by definition $s_1 > s_i$).

As every term in $t_1(s_2 + \cdots + s_\alpha)$ is a multiple of t_1, we can use g to eliminate each of the terms $t_1 s_\alpha$, $t_1 s_{\alpha-1}$, ..., $t_1 s_2$ in Equation (2.3) in turn:

$$\begin{aligned} & t_1(s_2 + \cdots + s_\alpha) - s_1(t_2 + \cdots + t_\beta) \\ \to\ & t_1(s_2 + \cdots + s_\alpha) - s_1(t_2 + \cdots + t_\beta) - s_\alpha g \\ =\ & t_1(s_2 + \cdots + s_{\alpha-1}) - s_1(t_2 + \cdots + t_\beta) - s_\alpha(t_2 + \cdots + t_\beta) \\ \to\ & t_1(s_2 + \cdots + s_{\alpha-2}) - (s_1 + s_{\alpha-1} + s_\alpha)(t_2 + \cdots + t_\beta) \\ \vdots\ & \\ \to\ & -(s_1 + s_2 + \cdots + s_\alpha)(t_2 + \cdots + t_\beta) \\ =\ & -s_1(t_2 + \cdots + t_\beta) - \cdots - s_\alpha(t_2 + \cdots + t_\beta). \end{aligned} \tag{2.4}$$

We do this in reverse order because, having eliminated a term $t_1 s_\gamma$ (where $3 \leqslant \gamma \leqslant \alpha$), to continue the term $t_1 s_{\gamma-1}$ must appear in the reduced polynomial (which it does because $t_1 s_{\gamma-1} > s_\delta t_\eta$ for all $\gamma \leqslant \delta \leqslant \alpha$ and $2 \leqslant \eta \leqslant \beta$).

We now use the same argument on $-s_1(t_2 + \cdots + t_\beta)$, using f to eliminate each of its terms in turn, giving the following reduction sequence.

$$\begin{array}{cl} & -s_1(t_2+\cdots+t_\beta) - \cdots - s_\alpha(t_2+\cdots+t_\beta) \\ \to & -s_1(t_2+\cdots+t_\beta) - \cdots - s_\alpha(t_2+\cdots+t_\beta) + t_2 f \\ = & -s_1(t_2+\cdots+t_\beta) - \cdots - s_\alpha(t_2+\cdots+t_\beta) + t_2(s_1+\cdots+s_\alpha) \\ = & -s_1(t_3+\cdots+t_\beta) - \cdots - s_\alpha(t_3+\cdots+t_\beta) \\ \to & -s_1(t_4+\cdots+t_\beta) - \cdots - s_\alpha(t_4+\cdots+t_\beta) \\ \vdots & \\ \to & 0. \end{array}$$

Technical point: If some term $s_i t_j$ (for $i, j \geqslant 2$) cancels the term $s_1 t_k$ (for $k \geqslant 3$) in Equation (2.4), then as we must have $j < k$ in order to have $s_i t_j = s_1 t_k$, the term $s_1 t_k$ will reappear as $s_i t_j$ when the term $s_1 t_j$ is eliminated, allowing us to continue the reduction as shown. This argument can be extended to the case where a combination of terms of the form $s_i t_j$ cancel the term $s_1 t_k$, as the term $s_1 t_k$ will reappear after all the terms $s_1 t_\kappa$ (for $2 \leqslant \kappa < k$) have been eliminated. $\square$

Proposition 2.5.2 (Buchberger's Second Criterion) *Let f, g and h be three members of a finite set of polynomials P over a commutative polynomial ring satisfying the following conditions.*

(a) $\mathrm{LM}(h) \mid \mathrm{lcm}(\mathrm{LM}(f), \mathrm{LM}(g))$.

(b) $\text{S-pol}(f, h) \to_P 0$ *and* $\text{S-pol}(g, h) \to_P 0$.

Then $\text{S-pol}(f, g) \to_P 0$.

Proof: If $\mathrm{LM}(h) \mid \mathrm{lcm}(\mathrm{LM}(f), \mathrm{LM}(g))$, then $m_h\mathrm{LM}(h) = \mathrm{lcm}(\mathrm{LM}(f), \mathrm{LM}(g))$ for some monomial m_h. Assume that $\mathrm{lcm}(\mathrm{LM}(f), \mathrm{LM}(g)) = m_f\mathrm{LM}(f) = m_g\mathrm{LM}(g)$ for some monomials m_f and m_g. Then it is clear that $m_f\mathrm{LM}(f) = m_h\mathrm{LM}(h)$ is a common multiple of $\mathrm{LM}(f)$ and $\mathrm{LM}(h)$, and $m_g\mathrm{LM}(g) = m_h\mathrm{LM}(h)$ is a common multiple of $\mathrm{LM}(g)$ and $\mathrm{LM}(h)$. It follows that $\mathrm{lcm}(\mathrm{LM}(f), \mathrm{LM}(g))$ is a multiple of both $\mathrm{lcm}(\mathrm{LM}(f), \mathrm{LM}(h))$ and $\mathrm{lcm}(\mathrm{LM}(g), \mathrm{LM}(h))$, so that

$$\mathrm{lcm}(\mathrm{LM}(f), \mathrm{LM}(g)) = m_{fh}\mathrm{lcm}(\mathrm{LM}(f), \mathrm{LM}(h)) = m_{gh}\mathrm{lcm}(\mathrm{LM}(g), \mathrm{LM}(h)) \qquad (2.5)$$

for some monomials m_{fh} and m_{gh}.

Because the S-polynomials S-pol(f, h) and S-pol(g, h) both reduce to zero using P, there are expressions

$$\text{S-pol}(f, h) - \sum_{i=1}^{\alpha} s_i p_i = 0$$

and

$$\text{S-pol}(g, h) - \sum_{j=1}^{\beta} t_j p_j = 0,$$

where the s_i and the t_j are terms, and $p_i, p_j \in P$ for all i and j. It follows that

$$\begin{aligned}
m_{fh}\left(\text{S-pol}(f, h) - \sum_{i=1}^{\alpha} s_i p_i\right) &= m_{gh}\left(\text{S-pol}(g, h) - \sum_{j=1}^{\beta} t_j p_j\right);\\
m_{fh}\left(\frac{\text{lcm}(\text{LM}(f), \text{LM}(h))}{\text{LT}(f)} f - \frac{\text{lcm}(\text{LM}(f), \text{LM}(h))}{\text{LT}(h)} h - \sum_{i=1}^{\alpha} s_i p_i\right) &= \\
m_{gh}\left(\frac{\text{lcm}(\text{LM}(g), \text{LM}(h))}{\text{LT}(g)} g - \frac{\text{lcm}(\text{LM}(g), \text{LM}(h))}{\text{LT}(h)} h - \sum_{j=1}^{\beta} t_j p_j\right);&\\
m_{fh}\left(\frac{\text{lcm}(\text{LM}(f), \text{LM}(g))}{m_{fh}\text{LT}(f)} f - \frac{\text{lcm}(\text{LM}(f), \text{LM}(g))}{m_{fh}\text{LT}(h)} h - \sum_{i=1}^{\alpha} s_i p_i\right) &= \\
m_{gh}\left(\frac{\text{lcm}(\text{LM}(f), \text{LM}(g))}{m_{gh}\text{LT}(g)} g - \frac{\text{lcm}(\text{LM}(f), \text{LM}(g))}{m_{gh}\text{LT}(h)} h - \sum_{j=1}^{\beta} t_j p_j\right);&\\
\frac{\text{lcm}(\text{LM}(f), \text{LM}(g))}{\text{LT}(f)} f - m_{fh}\sum_{i=1}^{\alpha} s_i p_i &= \frac{\text{lcm}(\text{LM}(f), \text{LM}(g))}{\text{LT}(g)} g - m_{gh}\sum_{j=1}^{\beta} t_j p_j;\\
\text{S-pol}(f, g) - \sum_{i=1}^{\alpha} m_{fh} s_i p_i + \sum_{j=1}^{\beta} m_{gh} t_j p_j &= 0.
\end{aligned}$$

To conclude that the S-polynomial S-pol(f, g) reduces to zero using P, it remains to show that the algebraic expression $-\sum_{i=1}^{\alpha} m_{fh} s_i p_i + \sum_{j=1}^{\beta} m_{gh} t_j p_j$ corresponds to a valid reduction of S-pol(f, g). To do this, it is sufficient to show that no term in either of the summations is greater than lcm$(\text{LM}(f), \text{LM}(g))$ (so that $\text{LM}(m_{fh} s_i p_i) < \text{lcm}(\text{LM}(f), \text{LM}(g))$ and $\text{LM}(m_{gh} t_j p_j) < \text{lcm}(\text{LM}(f), \text{LM}(g))$ for all i and j). But this follows from Equation (2.5) and from the fact that the original reductions of S-pol(f, h) and S-pol(g, h) are valid, so that $\text{LM}(s_i p_i) < \text{lcm}(\text{LM}(f), \text{LM}(h))$ and $\text{LM}(t_j p_j) < \text{lcm}(\text{LM}(g), \text{LM}(h))$ for all i and j. $\square$

2.5.2 Homogeneous Gröbner Bases

Definition 2.5.3 A polynomial is *homogeneous* if all its terms have the same degree. For example, the polynomial $x^2y+4yz^2+3z^3$ is homogeneous, but the polynomial x^3y+4x^2+45 is not homogeneous.

Of the many systems available for computing commutative Gröbner Bases, some (such as Bergman [6]) only admit sets of homogeneous polynomials as input. This restriction leads to gains in efficiency as we can take advantage of some of the properties of homogeneous polynomial arithmetic. For example, the S-polynomial of two homogeneous polynomials is homogeneous, and the reduction of a homogeneous polynomial by a set of homogeneous polynomials yields another homogeneous polynomial. It follows that if G is a Gröbner Basis for a set F of homogeneous polynomials, then G is another set of homogeneous polynomials.

At first glance, it seems that a system accepting only sets of homogeneous polynomials as input is not able to compute a Gröbner Basis for a set of polynomials containing one or more non-homogeneous polynomials. However, we can still use the system if we use an extendible monomial ordering and the processes of homogenisation and dehomogenisation.

Definition 2.5.4 Let $p = p_0 + \cdots + p_m$ be a polynomial over the polynomial ring $R[x_1, \ldots, x_n]$, where each p_i is the sum of the degree i terms in p (we assume that $p_m \neq 0$). The *homogenisation* of p with respect to a new (homogenising) variable y is the polynomial

$$h(p) := p_0y^m + p_1y^{m-1} + \cdots + p_{m-1}y + p_m,$$

where $h(p)$ belongs to a polynomial ring determined by where y is placed in the lexicographical ordering of the variables.

Definition 2.5.5 The *dehomogenisation* of a polynomial p is the polynomial $d(p)$ given by substituting $y = 1$ in p, where y is the homogenising variable. For example, the dehomogenisation of the polynomial $x_1^3 + x_1x_2y + x_1y^2 \in \mathbb{Q}[x_1, x_2, y]$ is the polynomial $x_1^3 + x_1x_2 + x_1 \in \mathbb{Q}[x_1, x_2]$.

Definition 2.5.6 A monomial ordering O is *extendible* if, given any polynomial $p = t_1 + \cdots + t_\alpha$ ordered with respect to O (where $t_1 > \cdots > t_\alpha$), the homogenisation of p preserves the order on the terms ($t'_i > t'_{i+1}$ for all $1 \leqslant i \leqslant \alpha - 1$, where the homogenisation

process maps the term $t_i \in p$ to the term $t'_i \in h(p)$).

Of the monomial orderings defined in Section 1.2.1, two of them (Lex and DegRevLex) are extendible as long as we ensure that the new variable y is lexicographically *less* than any of the variables $x_1, \ldots, x_n$; another (InvLex) is extendible as long as we ensure that the new variable y is lexicographically *greater* than any of the variables $x_1, \ldots, x_n$.

The other monomial orderings are *not* extendible as, no matter where we place the new variable y in the ordering of the variables, we can always find two monomials m_1 and m_2 such that, if $p = m_1 + m_2$ (with $m_1 > m_2$), then in $h(p) = m'_1 + m'_2$, we have $m'_1 < m'_2$. For example, $m_1 := x_1 x_2^2$ and $m_2 := x_1^2$ provides a counterexample for the DegLex monomial ordering.

Definition 2.5.7 Let $F = \{f_1, \ldots, f_m\}$ be a non-homogeneous set of polynomials. To compute a Gröbner Basis for F using a program that only accepts sets of homogeneous polynomials as input, we proceed as follows.

(a) Construct a homogeneous set of polynomials $F' = \{h(f_1), \ldots, h(f_m)\}$.

(b) Compute a Gröbner Basis G' for F'.

(c) Dehomogenise each polynomial $g' \in G'$ to obtain a set of polynomials G.

As long as the chosen monomial ordering O is extendible, G will be a Gröbner Basis for F with respect to O [22, page 113]. A word of warning however – this process is not necessarily more efficient that the direct computation of a Gröbner Basis for F using a program that does accept non-homogeneous sets of polynomials as input.

2.5.3 Selection Strategies

One of the most important factors when considering the efficiency of Buchberger's algorithm is the order in which S-polynomials are processed during the algorithm. A particular choice of a *selection strategy* to use can often cut down substantially the amount of work required in order to obtain a particular Gröbner Basis.

In 1979, Buchberger defined the *normal strategy* [10] that chooses to process an S-polynomial S-pol(f, g) if the monomial $\text{lcm}(\text{LM}(f), \text{LM}(g))$ is minimal (in the chosen

monomial ordering) amongst all such lowest common multiples. This strategy was refined in 1991 to give the *sugar strategy* [29], a strategy that chooses an S-polynomial to process if the *sugar* of the S-polynomial (a value associated to the S-polynomial) is minimal amongst all such values (the normal strategy is used in the event of a tie).

Motivation for the sugar strategy comes from the observation that the normal strategy performs well when used with a degree-based monomial ordering and a homogeneous basis; the sugar strategy was developed as a way to proceed based on what would happen when using the normal strategy in the computation of a Gröbner Basis for the corresponding homogenised input basis. We can therefore think of the sugar of an S-polynomial as representing the degree of the corresponding S-polynomial in the homogeneous computation.

The sugar of an S-polynomial is computed by using the following rules on the sugars of polynomials we encounter during the computation of a Gröbner Basis for the set of polynomials $F = \{f_1, \ldots, f_m\}$.

(1) The sugar Sug_{f_i} of a polynomial $f_i \in F$ is the total degree of the polynomial f_i (which is the degree of the term of maximal degree in f_i).

(2) If p is a polynomial and if t is a term, then $\mathrm{Sug}_{tp} = \deg(t) + \mathrm{Sug}_p$.

(3) If $p = p_1 + p_2$, then $\mathrm{Sug}_p = \max(\mathrm{Sug}_{p_1}, \mathrm{Sug}_{p_2})$.

It follows that the sugar of the S-polynomial $\text{S-pol}(g, h) = \frac{\mathrm{lcm}(\mathrm{LM}(g),\mathrm{LM}(h))}{\mathrm{LT}(g)} g - \frac{\mathrm{lcm}(\mathrm{LM}(g),\mathrm{LM}(h))}{\mathrm{LT}(h)} h$ is given by the formula

$$\mathrm{Sug}_{\text{S-pol}(g,h)} = \max(\mathrm{Sug}_g - \deg(\mathrm{LM}(g)), \mathrm{Sug}_h - \deg(\mathrm{LM}(h))) + \deg(\mathrm{lcm}(\mathrm{LM}(g), \mathrm{LM}(h))).$$

Example 2.5.8 To illustrate how a selection strategy reduces the amount of work required to compute a Gröbner Basis, consider the ideal generated by the basis $\{x^{31} - x^6 - x - y,\ x^8 - z,\ x^{10} - t\}$ over the polynomial ring $\mathbb{Q}[x, y, z, t]$. In our own implementation of Buchberger's algorithm, here is the number of S-polynomials processed during the algorithm when different selection strategies and different monomial orderings are used (the numbers quoted take into account the application of both of Buchberger's criteria).

Selection Strategy	Lex	DegLex	DegRevLex
No strategy	640	275	320
Normal strategy	123	63	61
Sugar strategy	96	55	54

2.5.4 Basis Conversion Algorithms

One factor which heavily influences the amount of time taken to compute a Gröbner Basis is the monomial ordering chosen. It is well known that some monomial orderings (such as Lex) are characterised as being 'slow', while other monomial orderings (such as DegRevLex) are said to be 'fast'. In practice what this means is that it usually takes far more time to calculate (say) a Lex Gröbner Basis than it does to calculate a DegRevLex Gröbner Basis for the same generating set of polynomials.

Because many of the useful applications of Gröbner Bases (such as solving systems of polynomial equations) depend on using 'slow' monomial orderings, a number of algorithms were developed in the 1990's that allow us to obtain a Gröbner Basis with respect to one monomial ordering from a Gröbner Basis with respect to another monomial ordering.

The idea is that the time it takes to compute a Gröbner Basis with respect to a 'fast' monomial ordering and then to convert it to a Gröbner Basis with respect to a 'slow' monomial ordering may be significantly less than the time it takes to compute a Gröbner Basis for the 'slow' monomial ordering directly. Although seemingly counterintuitive, the idea works well in practice.

One of the first conversion methods developed was the *FGLM* method, named after the four authors who published the paper [21] introducing it. The method relies on linear algebra to do the conversion, working with coefficient matrices and irreducible monomials. Its only drawback lies in the fact that it can only be used with zero-dimensional ideals, which are the ideals containing only a finite number of irreducible monomials (for each variable x_i in the polynomial ring, a Gröbner Basis for a zero-dimensional ideal must contain a polynomial which has a power of x_i as the leading monomial). This restriction does not apply in the case of the *Gröbner Walk* [18], a basis conversion method we shall study in further detail in Chapter 6.

2.5.5 Optimal Variable Orderings

In many cases, the ordering of the variables in a polynomial ring can have a significant effect on the time it takes to compute a Gröbner Basis for a particular ideal (an example can be found in [17]). This is worth bearing in mind if we are searching for *any* Gröbner Basis with respect to a certain ideal, so do not mind which variable ordering is being used. A heuristically optimal variable ordering is described in [34] (deriving from a discussion in [9]), where we order the variables so that the variable that occurs least often in the polynomials of the input basis is the largest variable; the second least common variable is the second largest variable; and so on (ties are broken randomly).

Example 2.5.9 Let $F := \{y^2z^2+x^2y,\ x^2y^4z+xy^2z+y^3,\ y^7+x^3z\}$ generate an ideal over the polynomial ring $\mathbb{Q}[x, y, z]$. Because x occurs 8 times in F, y occurs 19 times and z occurs 5 times, the heuristically optimal variable ordering is $z > x > y$. This is supported by the following table showing the times taken to compute a Lex Gröbner Basis for F using all six possible variable orderings, where we see that the time for the heuristically optimal variable ordering is close to the time for the true optimal variable ordering.

Variable Order	Time	Size of Gröbner Basis
$x > y > z$	1:15.10	6
$x > z > y$	0:02.85	7
$y > x > z$	2:19.45	7
$y > z > x$	2:16.09	7
$z > x > y$	0:05.91	8
$z > y > x$	5:44.38	8

2.5.6 Logged Gröbner Bases

In some situations, such as in the algorithm for the Gröbner Walk, it is desirable to be able to express each member of a Gröbner Basis in terms of members of the original basis from which the Gröbner Basis was computed. When we have such representations, our Gröbner Basis is said to be a *Logged Gröbner Basis*.

Definition 2.5.10 Let $G = \{g_1, \dots, g_p\}$ be a Gröbner Basis computed from an initial basis $F = \{f_1, \dots, f_m\}$. We say that G is a *Logged Gröbner Basis* if, for each $g_i \in G$, we

have an explicit expression of the form

$$g_i = \sum_{\alpha=1}^{\beta} t_\alpha f_{k_\alpha},$$

where the t_α are terms and $f_{k_\alpha} \in F$ for all $1 \leqslant \alpha \leqslant \beta$.

Proposition 2.5.11 *Given a finite basis $F = \{f_1, \ldots, f_m\}$, it is always possible to compute a Logged Gröbner Basis for F.*

Proof: We are required to prove that every polynomial added to the input basis $F = \{f_1, \ldots, f_m\}$ during Buchberger's algorithm has a representation in terms of members of F. But any such polynomial must be a reduced S-polynomial, so it follows that the first polynomial f_{m+1} added to F will always have the form

$$f_{m+1} = \text{S-pol}(f_i, f_j) - \sum_{\alpha=1}^{\beta} t_\alpha f_{k_\alpha},$$

where $f_i, f_j, f_{k_\alpha} \in F$ and the t_α are terms. This expression clearly gives a representation of our new polynomial in terms of members of F, and by induction (using substitution) it is also clear that each subsequent polynomial added to F will also have a representation in terms of members of F. □

Example 2.5.12 Let $F := \{f_1, f_2, f_3\} = \{xy - z,\ 2x + yz + z,\ x + yz\}$ generate an ideal over the polynomial ring $\mathbb{Q}[x, y, z]$, and let the monomial ordering be Lex. In obtaining a Gröbner Basis for F using Buchberger's algorithm, three new polynomials are added to F, giving a Gröbner Basis $G := \{g_1, g_2, g_3, g_4, g_5, g_6\} = \{xy - z,\ 2x + yz + z,\ x+yz,\ -\frac{1}{2}yz+\frac{1}{2}z,\ -2z^2,\ -2z\}$. These three new polynomials are obtained from the S-polynomials S-pol$(2x+yz+z, x+yz)$, S-pol$(xy-z, -\frac{1}{2}yz+\frac{1}{2}z)$ and S-pol$(xy-z, 2x+yz+z)$

respectively:

$$\begin{aligned}
\text{S-pol}(2x+yz+z,\ x+yz) &= \frac{1}{2}(2x+yz+z)-(x+yz) \\
&= -\frac{1}{2}yz+\frac{1}{2}z; \\
\text{S-pol}\left(xy-z,\ -\frac{1}{2}yz+\frac{1}{2}z\right) &= z(xy-z)+2x\left(-\frac{1}{2}yz+\frac{1}{2}z\right) \\
&= xz-z^2 \\
&\to_{f_2} xz-z^2-\frac{1}{2}z(2x+yz+z) \\
&= -\frac{1}{2}yz^2-\frac{3}{2}z^2 \\
&\to_{g_4} -\frac{1}{2}yz^2-\frac{3}{2}z^2-z\left(-\frac{1}{2}yz+\frac{1}{2}z\right) \\
&= -2z^2; \\
\text{S-pol}(xy-z,\ 2x+yz+z) &= (xy-z)-\frac{1}{2}y(2x+yz+z) \\
&= -\frac{1}{2}y^2z-\frac{1}{2}yz-z \\
&\to_{g_4} -\frac{1}{2}y^2z-\frac{1}{2}yz-z-y\left(-\frac{1}{2}yz+\frac{1}{2}z\right) \\
&= -yz-z \\
&\to_{g_4} -yz-z-2\left(-\frac{1}{2}yz+\frac{1}{2}z\right) \\
&= -2z.
\end{aligned}$$

These reductions enable us to give the following Logged Gröbner Basis for F.

Member of G	Logged Representation
$g_1 = xy-z$	f_1
$g_2 = 2x+yz+z$	f_2
$g_3 = x+yz$	f_3
$g_4 = -\frac{1}{2}yz+\frac{1}{2}z$	$\frac{1}{2}f_2-f_3$
$g_5 = -2z^2$	$zf_1+(x-z)f_2+(-2x+z)f_3$
$g_6 = -z$	$f_1+(-y-1)f_2+(y+2)f_3$

Chapter 3

Noncommutative Gröbner Bases

Once the potential of Gröbner Basis theory started to be realised in the 1970's, it was only natural to try to generalise the theory to related areas such as noncommutative polynomial rings. In 1986, Teo Mora published a paper [45] giving an algorithm for constructing a noncommutative Gröbner Basis. This work built upon the work of George Bergman; in particular his "diamond lemma for ring theory" [8].

In this chapter, we will describe Mora's algorithm and the theory behind it, in many ways giving a 'noncommutative version' of the previous chapter. This means that some material from the previous chapter will be duplicated; this however will be justified when the subtle differences between the cases becomes apparent, differences that are all too often overlooked when an 'easy generalisation' is made!

As in the previous chapter, we will consider the theory from the point of view of S-polynomials, in particular defining a noncommutative Gröbner Basis as a set of polynomials for which the S-polynomials all reduce to zero. At the end of the chapter, in order to give a flavour of a noncommutative Gröbner Basis program, we will give an extended example of the computation of a noncommutative Gröbner Basis, taking advantage of some of the improvements to Mora's algorithm such as Buchberger's criteria and selection strategies.

3.1 Overlaps

For a (two-sided) ideal J over a noncommutative polynomial ring, the concept of a Gröbner Basis for J remains the same: it is a set of polynomials G generating J such that remainders with respect to G are unique. How we obtain that Gröbner Basis also remains the same (we add S-polynomials to an initial basis as required); the difference comes in the definition of an S-polynomial.

Recall (from Section 2.1) that the purpose of an S-polynomial $\text{S-pol}(p_1, p_2)$ is to ensure that any polynomial p reducible by both p_1 and p_2 has a unique remainder when divided by a set of polynomials containing p_1 and p_2. In the commutative case, there is only one way to divide p by p_1 or p_2 (giving reductions $p - t_1p_1$ or $p - t_2p_2$ respectively, where t_1 and t_2 are terms); this means that there is only one S-polynomial for each pair of polynomials. In the noncommutative case however, a polynomial may divide another polynomial in many different ways (for example the polynomial $xyx - z$ divides the polynomial $xyxyx + 4x^2$ in two different ways, giving reductions $zyx + 4x^2$ and $xyz + 4x^2$). For this reason, we do not have a fixed number of S-polynomials for each pair (p_1, p_2) of polynomials in the noncommutative case – that number will depend on the number of *overlaps* between the lead monomials of p_1 and p_2.

In order to explain what an overlap is, we first need the following preliminary definitions allowing us to select a particular part of a noncommutative monomial.

Definition 3.1.1 Consider a monomial m of degree d over a noncommutative polynomial ring $\mathcal{R}$.

- Let $\text{Prefix}(m, i)$ denote the prefix of m of degree i (where $1 \leqslant i \leqslant d$). For example, $\text{Prefix}(x^2yz, 3) = x^2y$; $\text{Prefix}(zyx^2, 1) = z$ and $\text{Prefix}(y^2zx, 4) = y^2zx$.
- Let $\text{Suffix}(m, i)$ denote the suffix of m of degree i (where $1 \leqslant i \leqslant d$). For example, $\text{Suffix}(x^2yz, 3) = xyz$; $\text{Suffix}(zyx^2, 1) = x$ and $\text{Suffix}(y^2zx, 4) = y^2zx$.
- Let $\text{Subword}(m, i, j)$ denote the subword of m starting at position i and finishing at position j (where $1 \leqslant i \leqslant j \leqslant d$). For example, $\text{Subword}(zyx^2, 2, 3) = yx$; $\text{Subword}(zyx^2, 3, 3) = x$ and $\text{Subword}(y^2zx, 1, 4) = y^2zx$.

Definition 3.1.2 Let m_1 and m_2 be two monomials over a noncommutative polynomial ring $\mathcal{R}$ with respective degrees $d_1 \geqslant d_2$. We say that m_1 and m_2 *overlap* if any of the

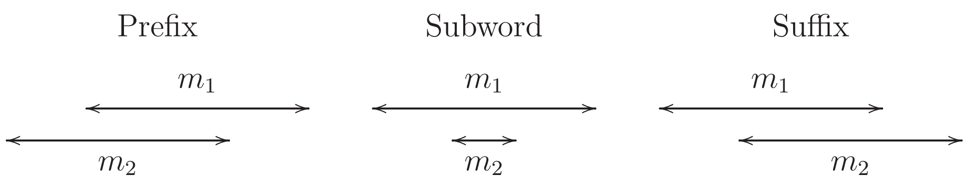
following conditions are satisfied.

(a) $\text{Prefix}(m_1, i) = \text{Suffix}(m_2, i)$ $(1 \leqslant i < d_2)$;

(b) $\text{Subword}(m_1, i, i + d_2 - 1) = m_2$ $(1 \leqslant i \leqslant d_1 - d_2 + 1)$;

(c) $\text{Suffix}(m_1, i) = \text{Prefix}(m_2, i)$ $(1 \leqslant i < d_2)$.

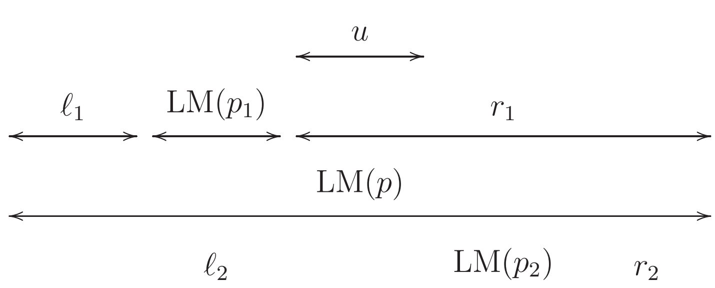
We will refer to the above overlap types as being prefix, subword and suffix overlaps respectively; we can picture the overlap types as follows.



Remark 3.1.3 We have defined the cases where m_2 is a prefix or a suffix of m_1 to be subword overlaps.

Proposition 3.1.4 *Let p be a polynomial over a noncommutative polynomial ring $\mathcal{R}$ that is divisible by two polynomials $p_1, p_2 \in \mathcal{R}$, so that $\ell_1 \text{LM}(p_1) r_1 = \text{LM}(p) = \ell_2 \text{LM}(p_2) r_2$ for some monomials ℓ_1, ℓ_2, r_1, r_2. As positioned in $\text{LM}(p)$, if $\text{LM}(p_1)$ and $\text{LM}(p_2)$ do not overlap, then no matter which of the two reductions of p we apply first, we can always obtain a common remainder.*

Proof: We picture the situation as follows (u is a monomial).



We construct the common remainder by using p_2 to divide the remainder we obtain by dividing p by p_1 (and vice versa).

Reduction by p_1 first

$$\begin{array}{rcl} p & \rightarrow & p - (\mathrm{LC}(p)\mathrm{LC}(p_1)^{-1})\ell_1 p_1 r_1 \\ & = & (p - \mathrm{LT}(p)) - (\mathrm{LC}(p)\mathrm{LC}(p_1)^{-1})\ell_1(p_1 - \mathrm{LT}(p_1))r_1 \\ & = & (p - \mathrm{LT}(p)) - (\mathrm{LC}(p)\mathrm{LC}(p_1)^{-1})\ell_1(p_1 - \mathrm{LT}(p_1))u\mathrm{LM}(p_2)r_2 \\ & \xrightarrow{*} & (p - \mathrm{LT}(p)) - (\mathrm{LC}(p)\mathrm{LC}(p_1)^{-1}\mathrm{LC}(p_2)^{-1})\ell_1(p_1 - \mathrm{LT}(p_1))u(p_2 - \mathrm{LT}(p_2))r_2 \end{array}$$

Reduction by p_2 first

$$\begin{array}{rcl} p & \rightarrow & p - (\mathrm{LC}(p)\mathrm{LC}(p_2)^{-1})\ell_2 p_2 r_2 \\ & = & (p - \mathrm{LT}(p)) - (\mathrm{LC}(p)\mathrm{LC}(p_2)^{-1})\ell_2(p_2 - \mathrm{LT}(p_2))r_2 \\ & = & (p - \mathrm{LT}(p)) - (\mathrm{LC}(p)\mathrm{LC}(p_2)^{-1})\ell_1\mathrm{LM}(p_1)u(p_2 - \mathrm{LT}(p_2))r_2 \\ & \xrightarrow{*} & (p - \mathrm{LT}(p)) - (\mathrm{LC}(p)\mathrm{LC}(p_1)^{-1}\mathrm{LC}(p_2)^{-1})\ell_1(p_1 - \mathrm{LT}(p_1))u(p_2 - \mathrm{LT}(p_2))r_2 \end{array}$$

□

Let p, p_1, p_2, ℓ_1, ℓ_2, r_1 and r_2 be as in Proposition 3.1.4. As positioned in $\mathrm{LM}(p)$, in general the lead monomials of p_1 and p_2 may or may not overlap, giving four different possibilities, each of which is illustrated by an example in the following table.

$\mathrm{LM}(p)$	ℓ_1	$\mathrm{LM}(p_1)$	r_1	ℓ_2	$\mathrm{LM}(p_2)$	r_2	Overlap?
x^2yzxy^3	x^2yz	xy^3	1	x^2y	zx	y^3	Prefix overlap
x^2yzxy^3	x	$xyzxy$	y^2	x^2	yzx	y^3	Subword overlap
x^2yzxy^3	x	xyz	xy^3	x^2y	zx	y^3	Suffix overlap
x^2yzxy^3	x^2	y	zxy^3	x^2yz	xy^2	y	No overlap

In the cases that $\mathrm{LM}(p_1)$ and $\mathrm{LM}(p_2)$ do overlap, we are not guaranteed to be able to obtain a common remainder when we divide p by both p_1 and p_2. To counter this, we introduce (as in the commutative case) an S-polynomial into our dividing set to ensure a common remainder, requiring one S-polynomial for every possible way that $\mathrm{LM}(p_1)$ and $\mathrm{LM}(p_2)$ overlap, including *self overlaps* (where $p_1 = p_2$, for example $\mathrm{Prefix}(xyx, 1) = \mathrm{Suffix}(xyx, 1)$).

Definition 3.1.5 Let the lead monomials of two polynomials p_1 and p_2 overlap in such a way that $\ell_1\mathrm{LM}(p_1)r_1 = \ell_2\mathrm{LM}(p_2)r_2$, where ℓ_1, ℓ_2, r_1 and r_2 are monomials chosen so that at least one of ℓ_1 and ℓ_2 and at least one of r_1 and r_2 is equal to the unit monomial. The *S-polynomial* associated with this overlap is given by the expression

$$\text{S-pol}(\ell_1, p_1, \ell_2, p_2) = c_1\ell_1 p_1 r_1 - c_2\ell_2 p_2 r_2,$$

where $c_1 = \mathrm{LC}(p_2)$ and $c_2 = \mathrm{LC}(p_1)$.

Remark 3.1.6 The monomials ℓ_1 and ℓ_2 are included in the notation S-pol$(\ell_1, p_1, \ell_2, p_2)$ in order to differentiate between distinct S-polynomials involving the two polynomials p_1 and p_2 (there is no need to include r_1 and r_2 in the notation because r_1 and r_2 are uniquely determined by ℓ_1 and ℓ_2 respectively).

Example 3.1.7 Consider the polynomial $p := xyz + 2y$ and the set of polynomials $P := \{p_1, p_2\} = \{xy - z, yz - x\}$, all polynomials being ordered by DegLex and originating from the polynomial ring $\mathbb{Q}\langle x, y, z\rangle$. We see that p is divisible (in one way) by both of the polynomials in P, giving remainders $z^2 + 2y$ and $x^2 + 2y$ respectively, both of which are irreducible by P. It follows that p does not have a unique remainder with respect to P.

Because there is only one overlap involving the lead monomials of p_1 and p_2, namely Suffix$(xy, 1)$ = Prefix$(yz, 1)$, there is only one S-polynomial for the set P, which is the polynomial $(xy - z)z - x(yz - x) = x^2 - z^2$. When we add this polynomial to the set P, we see that the remainder of p with respect to the enlarged P is now unique, as the remainder of p with respect to p_2 (the polynomial $x^2 + 2y$) is now reducible by our new polynomial, giving a new remainder $z^2 + 2y$ which agrees with the remainder of p with respect to p_1.

Let us now give a definition of a noncommutative Gröbner Basis in terms of S-polynomials.

Definition 3.1.8 Let $G = \{g_1, \ldots, g_m\}$ be a basis for an ideal J over a noncommutative polynomial ring $\mathcal{R} = R\langle x_1, \ldots, x_n\rangle$. If all the S-polynomials involving members of G reduce to zero using G, then G is a *noncommutative Gröbner Basis* for J.

Theorem 3.1.9 *Given any polynomial p over a polynomial ring $\mathcal{R} = R\langle x_1, \ldots, x_n\rangle$, the remainder of the division of p by a basis G for an ideal J in $\mathcal{R}$ is unique if and only if G is a Gröbner Basis.*

Proof: ($\Rightarrow$) Following the proof of Theorem 2.1.5, we need to show that the division process is *locally confluent*, that is if there are polynomials f, f_1, $f_2 \in \mathcal{R}$ with $f_1 = f - \ell_1 g_1 r_1$ and $f_2 = f - \ell_2 g_2 r_2$ for terms ℓ_1, ℓ_2, r_1, r_2 and $g_1, g_2 \in G$, then there exists a polynomial $f_3 \in \mathcal{R}$ such that both f_1 and f_2 reduce to f_3. As before, this is equivalent to showing that the polynomial $f_2 - f_1 = \ell_1 g_1 r_1 - \ell_2 g_2 r_2$ reduces to zero.

If $\mathrm{LT}(\ell_1 g_1 r_1) \neq \mathrm{LT}(\ell_2 g_2 r_2)$, then the remainders f_1 and f_2 are obtained by cancelling off different terms of the original f (the reductions of f are *disjoint*), so it is possible, assuming (without loss of generality) that $\mathrm{LT}(\ell_1 g_1 r_1) > \mathrm{LT}(\ell_2 g_2 r_2)$, to directly reduce the polynomial $f_2 - f_1 = \ell_1 g_1 r_1 - \ell_2 g_2 r_2$ in the following manner: $\ell_1 g_1 r_1 - \ell_2 g_2 r_1 \rightarrow_{g_1} -\ell_2 g_2 r_2 \rightarrow_{g_2} 0$.

On the other hand, if $\mathrm{LT}(\ell_1 g_1 r_1) = \mathrm{LT}(\ell_2 g_2 r_2)$, then the reductions of f are not disjoint (as the same term t from f is cancelled off during both reductions), so that the term t does not appear in the polynomial $\ell_1 g_1 r_1 - \ell_2 g_2 r_2$. However, the monomial $\mathrm{LM}(t)$ must contain the monomials $\mathrm{LM}(g_1)$ and $\mathrm{LM}(g_2)$ as subwords if both g_1 and g_2 cancel off the term t, so it follows that $\mathrm{LM}(g_1)$ and $\mathrm{LM}(g_2)$ will either overlap or not overlap in $\mathrm{LM}(t)$. If they do not overlap, then we know from Proposition 3.1.4 that f_1 and f_2 will have a common remainder ($f_1 \xrightarrow{*} f_3$ and $f_2 \xrightarrow{*} f_3$), so that $f_2 - f_1 \xrightarrow{*} f_3 - f_3 = 0$. Otherwise, because of the overlap between $\mathrm{LM}(g_1)$ and $\mathrm{LM}(g_2)$, the polynomial $\ell_1 g_1 r_1 - \ell_2 g_2 r_2$ will be a multiple of an S-polynomial, say $\ell_1 g_1 r_1 - \ell_2 g_2 r_2 = \ell_3(\text{S-pol}(\ell_1', g_1, \ell_2', g_2))r_3$ for some terms ℓ_3, r_3 and some monomials ℓ_1', ℓ_2'. But G is a Gröbner Basis, so the S-polynomial $\text{S-pol}(\ell_1', g_1, \ell_2', g_2)$ will reduce to zero, and hence by extension the polynomial $\ell_1 g_1 r_1 - \ell_2 g_2 r_2$ will also reduce to zero.

($\Leftarrow$) As all S-polynomials are members of the ideal J, to complete the proof it is sufficient to show that there is always a reduction path of an arbitrary member of the ideal that leads to a zero remainder (the uniqueness of remainders will then imply that members of the ideal always reduce to zero). Let $f \in J = \langle G \rangle$. Then, by definition, there exist $g_i \in G$ (not necessarily all different) and terms $\ell_i, r_i \in \mathcal{R}$ (where $1 \leqslant i \leqslant j$) such that

$$f = \sum_{i=1}^{j} \ell_i g_i r_i.$$

We proceed by induction on j. If $j = 1$, then $f = \ell_1 g_1 r_1$, and it is clear that we can use g_1 to reduce f to give a zero remainder ($f \rightarrow_{g_1} f - \ell_1 g_1 r_1 = 0$). Assume that the result is true for $j = k$, and let us look at the case $j = k + 1$, so that

$$f = \left(\sum_{i=1}^{k} \ell_i g_i r_i \right) + \ell_{k+1} g_{k+1} r_{k+1}.$$

By the inductive hypothesis, $\sum_{i=1}^{k} \ell_i g_i r_i$ is a member of the ideal that reduces to zero. The polynomial f therefore reduces to the polynomial $f' := \ell_{k+1} g_{k+1} r_{k+1}$, and we can

now use g_{k+1} to reduce f' to give a zero remainder ($f' \rightarrow_{g_{k+1}} f' - \ell_{k+1}g_{k+1}r_{k+1} = 0$). □

Remark 3.1.10 The above Theorem forms part of Bergman's Diamond Lemma [8, Theorem 1.2].

3.2 Mora's Algorithm

Let us now consider the following pseudo code representing Mora's algorithm for computing noncommutative Gröbner Bases [45].

Algorithm 5 Mora's Noncommutative Gröbner Basis Algorithm

Input: A Basis $F = \{f_1, f_2, \dots, f_m\}$ for an ideal J over a noncommutative polynomial ring $R\langle x_1, \dots x_n\rangle$; an admissible monomial ordering O.

Output: A Gröbner Basis $G = \{g_1, g_2, \dots, g_p\}$ for J (in the case of termination).

Let $G = F$ and let $A = \emptyset$;
For each pair of polynomials (g_i, g_j) in G ($i \leqslant j$), add an S-polynomial S-pol$(\ell_1, g_i, \ell_2, g_j)$ to A for each overlap $\ell_1\mathrm{LM}(g_i)r_1 = \ell_2\mathrm{LM}(g_j)r_2$ between the lead monomials of $\mathrm{LM}(g_i)$ and $\mathrm{LM}(g_j)$.
while (A is not empty) **do**
  Remove the first entry s_1 from A;
  $s_1' = \mathrm{Rem}(s_1, G)$;
  if ($s_1' \neq 0$) **then**
    Add s_1' to G and then (for all $g_i \in G$) add all the S-polynomials of the form S-pol$(\ell_1, g_i, \ell_2, s_1')$ to A;
  end if
end while
return G;

Structurally, Mora's algorithm is virtually identical to Buchberger's algorithm, in that we compute and reduce each S-polynomial in turn; we add a reduced S-polynomial to our basis if it does not reduce to zero; and we continue until all S-polynomials reduce to zero — exactly as in Algorithm 3. Despite this, there are major differences from an implementation standpoint, not least in the fact that noncommutative polynomials are much more difficult to handle on a computer; and noncommutative S-polynomials need more complicated data structures. This may explain why implementations of the noncommutative Gröbner Basis algorithm are currently sparser than those for the commutative algorithm;

and also why such implementations often impose restrictions on the problems that can be handled — Bergman [6] for instance only allows input bases which are homogeneous.

3.2.1 Termination

In the commutative case, Dickson's Lemma and Hilbert's Basis Theorem allow us to prove that Buchberger's algorithm always terminates for all possible inputs. It is a fact however that Mora's algorithm does not terminate for all possible inputs (so that an ideal may have an infinite Gröbner Basis in general) because there is no analogue of Dickson's Lemma for noncommutative monomial ideals.

Proposition 3.2.1 *Not all noncommutative monomial ideals are finitely generated.*

Proof: Assume to the contrary that all noncommutative monomial ideals are finitely generated, and consider an ascending chain of such ideals $J_1 \subseteq J_2 \subseteq \cdots$. By our assumption, the ideal $J = \cup J_i$ (for $i \geqslant 1$) will be finitely generated, which means that there must be some $k \geqslant 1$ such that $J_k = J_{k+1} = \cdots$. For a counterexample, let $\mathcal{R} = \mathbb{Q}\langle x, y\rangle$ be a noncommutative polynomial ring, and define J_i (for $i \geqslant 1$) to be the ideal in $\mathcal{R}$ generated by the set of monomials $\{xyx, xy^2x, \ldots, xy^ix\}$. Because no member of this set is a multiple of any other member of the set, it is clear that there cannot be a $k \geqslant 1$ such that $J_k = J_{k+1} = \cdots$ because $xy^{k+1}x \in J_{k+1}$ and $xy^{k+1}x \notin J_k$ for all $k \geqslant 1$. $\square$

Another way of explaining why Mora's algorithm does not terminate comes from considering the link between noncommutative Gröbner Bases and the Knuth-Bendix Critical Pairs Completion Algorithm for monoid rewrite systems [39], an algorithm that attempts to find a complete rewrite system for any given monoid presentation. Because Mora's algorithm can be used to emulate the Knuth-Bendix algorithm (for the details, see for example [33]), if we assume that Mora's algorithm always terminates, then we have found a way to solve the *word problem* for monoids (so that we can determine whether any word in a given monoid is equal to the identity word); this however contradicts the fact that the word problem is actually an unsolvable problem (so that it is impossible to define an algorithm that can tell whether two words in a given monoid are identical).

3.3 Reduced Gröbner Bases

Definition 3.3.1 Let $G = \{g_1, \ldots, g_p\}$ be a Gröbner Basis for an ideal over a polynomial ring $R\langle x_1, \ldots, x_n\rangle$. G is a *reduced* Gröbner Basis if the following conditions are satisfied.

(a) $\mathrm{LC}(g_i) = 1_R$ for all $g_i \in G$.

(b) No term in any polynomial $g_i \in G$ is divisible by any $\mathrm{LT}(g_j)$, $j \neq i$.

Theorem 3.3.2 *If there exists a Gröbner Basis G for an ideal J over a noncommutative polynomial ring, then J has a unique reduced Gröbner Basis.*

Proof: *Existence.* We claim that the following procedure transforms G into a reduced Gröbner Basis G'.

(i) Multiply each $g_i \in G$ by $\mathrm{LC}(g_i)^{-1}$.

(ii) Reduce each $g_i \in G$ by $G \setminus \{g_i\}$, removing from G all polynomials that reduce to zero.

It is clear that G' satisfies the conditions of Definition 3.3.1, so it remains to show that G' is a Gröbner Basis, which we shall do by showing that the application of each step of instruction (ii) above produces a basis which is still a Gröbner Basis.

Let $G = \{g_1, \ldots, g_p\}$ be a Gröbner Basis, and let g_i' be the reduction of an arbitrary $g_i \in G$ with respect to $G \setminus \{g_i\}$, carried out as follows (the ℓ_k and the r_k are terms).

$$g_i' = g_i - \sum_{k=1}^{\kappa} \ell_k g_{j_k} r_k. \tag{3.1}$$

Set $H = (G\setminus\{g_i\})\cup\{g_i'\}$ if $g_i' \neq 0$, and set $H = G\setminus\{g_i\}$ if $g_i' = 0$. As G is a Gröbner Basis, all S-polynomials involving elements of G reduce to zero using G, so there are expressions

$$c_b\ell_a g_a r_a - c_a\ell_b g_b r_b - \sum_{u=1}^{\mu} \ell_u g_{c_u} r_u = 0 \tag{3.2}$$

for every S-polynomial $\text{S-pol}(\ell_a, g_a, \ell_b, g_b) = c_b\ell_a g_a r_a - c_a\ell_b g_b r_b$, where $c_a = \mathrm{LC}(g_a)$; $c_b = \mathrm{LC}(g_b)$; the ℓ_u and the r_u are terms (for $1 \leqslant u \leqslant \mu$); and $g_a, g_b, g_{c_u} \in G$. To show that H is

a Gröbner Basis, we must show that all S-polynomials involving elements of H reduce to zero using H. For polynomials $g_a, g_b \in H$ not equal to g_i', we can reduce an S-polynomial of the form $\text{S-pol}(\ell_a, g_a, \ell_b, g_b)$ using the reduction shown in Equation (3.2), substituting for g_i from Equation (3.1) if any of the g_{c_u} in Equation (3.2) are equal to g_i. This gives a reduction to zero of $\text{S-pol}(\ell_a, g_a, \ell_b, g_b)$ in terms of elements of H.

If $g_i' = 0$, our proof is complete. Otherwise consider all S-polynomials $\text{S-pol}(\ell_i', g_i', \ell_b, g_b)$ involving the pair of polynomials (g_i', g_b), where $g_b \in G \setminus \{g_i\}$. We claim that there exists an S-polynomial $\text{S-pol}(\ell_1, g_i, \ell_2, g_b) = c_b\ell_1 g_i r_1 - c_i\ell_2 g_b r_2$ such that $\text{S-pol}(\ell_i', g_i', \ell_b, g_b) = c_b\ell_1 g_i' r_1 - c_i\ell_2 g_b r_2$. To prove this claim, it is sufficient to show that $\text{LT}(g_i) = \text{LT}(g_i')$. Assume for a contradiction that $\text{LT}(g_i) \neq \text{LT}(g_i')$. It follows that during the reduction of g_i we were able to reduce its lead term, so that $\text{LT}(g_i) = \ell\text{LT}(g_j)r$ for some terms ℓ and r and some $g_j \in G$. Because $\text{LM}(g_i - \ell g_j r) < \text{LM}(g_i)$, the polynomial $g_i - \ell g_j r$ must reduce to zero without using g_i, so that $g_i' = 0$, giving a contradiction.

It remains to show that $\text{S-pol}(\ell_i', g_i', \ell_b, g_b) \to_H 0$. We know that $\text{S-pol}(\ell_1, g_i, \ell_2, g_b) = c_b\ell_1 g_i r_1 - c_i\ell_2 g_b r_2 \to_G 0$, and Equation (3.2) tells us that $c_b\ell_1 g_i r_1 - c_i\ell_2 g_b r_2 - \sum_{u=1}^{\mu} \ell_u g_{c_u} r_u = 0$. Substituting for g_i from Equation (3.1), we obtain[1]

$$c_b\ell_1\left(g_i' + \sum_{k=1}^{\kappa} \ell_k g_{j_k} r_k\right) r_1 - c_i\ell_2 g_b r_2 - \sum_{u=1}^{\mu} \ell_u g_{c_u} r_u = 0$$

or

$$c_b\ell_1 g_i' r_1 - c_i\ell_2 g_b r_2 - \left(\sum_{u=1}^{\mu} \ell_u g_{c_u} r_u - \sum_{k=1}^{\kappa} c_b\ell_1\ell_k g_{j_k} r_k r_1\right) = 0,$$

which implies that $\text{S-pol}(\ell_i', g_i', \ell_b, g_b) \to_H 0$. The only other case to consider is the case of an S-polynomial coming from a self overlap involving $\text{LM}(g_i')$. But because we now know that $\text{LT}(g_i') = \text{LT}(g_i)$, we can use exactly the same argument as above to show that the S-polynomial $\text{S-pol}(\ell_1, g_i', \ell_2, g_i')$ reduces to zero using H because an S-polynomial of the form $\text{S-pol}(\ell_1, g_i, \ell_2, g_i)$ will exist.

Uniqueness. Assume for a contradiction that $G = \{g_1, \ldots, g_p\}$ and $H = \{h_1, \ldots, h_q\}$ are two reduced Gröbner Bases for an ideal J, with $G \neq H$. Let g_i be an arbitrary element from G (where $1 \leqslant i \leqslant p$). Because g_i is a member of the ideal, then g_i must reduce to zero using H (H is a Gröbner Basis). This means that there must exist a polynomial

[1] Substitutions for g_i may also occur in the summation $\sum_{u=1}^{\mu} \ell_u g_{c_u} r_u$; these substitutions have not been considered in the displayed formulae.

$h_j \in H$ such that $\mathrm{LT}(h_j) \mid \mathrm{LT}(g_i)$. If $\mathrm{LT}(h_j) \neq \mathrm{LT}(g_i)$, then $\ell \times \mathrm{LT}(h_j) \times r = \mathrm{LT}(g_i)$ for some monomials ℓ and r, at least one of which is not equal to the unit monomial. But h_j is also a member of the ideal, so it must reduce to zero using G. Therefore there exists a polynomial $g_k \in G$ such that $\mathrm{LT}(g_k) \mid \mathrm{LT}(h_j)$, which implies that $\mathrm{LT}(g_k) \mid \mathrm{LT}(g_i)$, with $k \neq i$. This contradicts condition (b) of Definition 3.3.1 so that G cannot be a reduced Gröbner Basis for J if $\mathrm{LT}(h_j) \neq \mathrm{LT}(g_i)$. From this we deduce that each $g_i \in G$ has a corresponding $h_j \in H$ such that $\mathrm{LT}(g_i) = \mathrm{LT}(h_j)$. Further, because G and H are assumed to be reduced Gröbner Bases, this is a one-to-one correspondence.

It remains to show that if $\mathrm{LT}(g_i) = \mathrm{LT}(h_j)$, then $g_i = h_j$. Assume for a contradiction that $g_i \neq h_j$ and consider the polynomial $g_i - h_j$. Without loss of generality, assume that $\mathrm{LM}(g_i - h_j)$ appears in g_i. Because $g_i - h_j$ is a member of the ideal, then there is a polynomial $g_k \in G$ such that $\mathrm{LT}(g_k) \mid \mathrm{LT}(g_i - h_j)$. But this again contradicts condition (b) of Definition 3.3.1, as we have shown that there is a term in g_i that is divisible by $\mathrm{LT}(g_k)$ for some $k \neq i$. It follows that G cannot be a reduced Gröbner Basis if $g_i \neq h_j$, which means that $G = H$ and therefore reduced Gröbner Bases are unique. $\square$

As in the commutative case, we may refine the procedure for finding a unique reduced Gröbner Basis (as given in the proof of Theorem 3.3.2) by removing from the Gröbner Basis all polynomials whose lead monomials are multiples of the lead monomials of other Gröbner Basis elements. This leads to the definition of Algorithm 6.

3.4 Improvements to Mora's Algorithm

In Section 2.5, we surveyed some of the numerous improvements of Buchberger's algorithm. Let us now demonstrate that many of these improvements can also be applied in the noncommutative case.

3.4.1 Buchberger's Criteria

In the commutative case, Buchberger's first criterion states that we can ignore any S-polynomial $\text{S-pol}(f, g)$ in which $\mathrm{lcm}(\mathrm{LM}(f), \mathrm{LM}(g)) = \mathrm{LM}(f)\mathrm{LM}(g)$. In the noncommutative case, this translates as saying that we can ignore any 'S-polynomial' $\text{S-pol}(\ell_1, f, \ell_2, g) = \mathrm{LC}(g)\ell_1 f r_1 - \mathrm{LC}(f)\ell_2 g r_2$ such that $\mathrm{LM}(f)$ and $\mathrm{LM}(g)$ do not overlap in the monomial $\ell_1\mathrm{LM}(f)r_1 = \ell_2\mathrm{LM}(g)r_2$. We can certainly show that such an 'S-polynomial' will reduce to zero by utilising Proposition 3.1.4, but we will never be able to use this result as, by

Algorithm 6 The Noncommutative Unique Reduced Gröbner Basis Algorithm

Input: A Gröbner Basis $G = \{g_1, g_2, \dots, g_m\}$ for an ideal J over a noncommutative polynomial ring $R\langle x_1, \dots x_n\rangle$; an admissible monomial ordering O.

Output: The unique reduced Gröbner Basis $G' = \{g'_1, g'_2, \dots, g'_p\}$ for J.

$G' = \emptyset$;
for each $g_i \in G$ **do**
  Multiply g_i by $\mathrm{LC}(g_i)^{-1}$;
  if $(\mathrm{LM}(g_i) = \ell\mathrm{LM}(g_j)r$ for some monomials ℓ, r and some $g_j \in G$ $(g_j \neq g_i))$ **then**
    $G = G \setminus \{g_i\}$;
  end if
end for
for each $g_i \in G$ **do**
  $g'_i = \mathrm{Rem}(g_i, (G \setminus \{g_i\}) \cup G')$;
  $G = G \setminus \{g_i\}$; $G' = G' \cup \{g'_i\}$;
end for
return G';

definition, an S-polynomial is only defined when we have an overlap between $\mathrm{LM}(f)$ and $\mathrm{LM}(g)$. It follows that an 'S-polynomial' of the above type will never occur in Mora's algorithm, and so Buchberger's first criterion is redundant in the noncommutative case. The same cannot be said of his second criterion however, which certainly does improve the efficiency of Mora's algorithm.

Proposition 3.4.1 (Buchberger's Second Criterion) *Let f, g and h be three members of a finite set of polynomials P over a noncommutative polynomial ring, and consider an S-polynomial of the form*

$$\text{S-pol}(\ell_1, f, \ell_2, g) = c_2\ell_1 f r_1 - c_1\ell_2 g r_2. \tag{3.3}$$

If $\mathrm{LM}(h) \mid \ell_1\mathrm{LM}(f)r_1$*, so that*

$$\ell_1\mathrm{LM}(f)r_1 = \ell_3\mathrm{LM}(h)r_3 = \ell_2\mathrm{LM}(g)r_2 \tag{3.4}$$

for some monomials ℓ_3, r_3, then $\text{S-pol}(\ell_1, f, \ell_2, g) \rightarrow_P 0$ *if all S-polynomials corresponding to overlaps (as placed in the monomial $\ell_1\mathrm{LM}(f)r_1$) between* $\mathrm{LM}(h)$ *and either* $\mathrm{LM}(f)$ *or* $\mathrm{LM}(g)$ *reduce to zero using P.*

Proof (cf. [37], Appendix A): To be able to describe an S-polynomial corresponding to an overlap (as placed in the monomial $\ell_1\mathrm{LM}(f)r_1$) between $\mathrm{LM}(h)$ and either $\mathrm{LM}(f)$ or $\mathrm{LM}(g)$, we introduce the following notation.

- Let ℓ_{13} be the monomial corresponding to the common prefix of ℓ_1 and ℓ_3 of maximal degree, so that $\ell_1 = \ell_{13}\ell_1'$ and $\ell_3 = \ell_{13}\ell_3'$. (Here, and similarly below, if there is no common prefix of ℓ_1 and ℓ_3, then $\ell_{13} = 1$, $\ell_1' = \ell_1$ and $\ell_3' = \ell_3$.)
- Let ℓ_{23} be the monomial corresponding to the common prefix of ℓ_2 and ℓ_3 of maximal degree, so that $\ell_2 = \ell_{23}\ell_2''$ and $\ell_3 = \ell_{23}\ell_3''$.
- Let r_{13} be the monomial corresponding to the common suffix of r_1 and r_3 of maximal degree, so that $r_1 = r_1'r_{13}$ and $r_3 = r_3'r_{13}$.
- Let r_{23} be the monomial corresponding to the common suffix of r_2 and r_3 of maximal degree, so that $r_2 = r_2''r_{23}$ and $r_3 = r_3''r_{23}$.

We can now manipulate Equation (3.3) as follows (where $c_3 = \mathrm{LC}(h)$).

$$\begin{aligned}
c_3(\text{S-pol}(\ell_1, f, \ell_2, g)) &= c_3c_2\ell_1fr_1 - c_3c_1\ell_2gr_2 \\
&= c_3c_2\ell_1fr_1 - c_1c_2\ell_3hr_3 + c_1c_2\ell_3hr_3 - c_3c_1\ell_2gr_2 \\
&= c_2(c_3\ell_1fr_1 - c_1\ell_3hr_3) - c_1(c_3\ell_2gr_2 - c_2\ell_3hr_3) \\
&= c_2(c_3\ell_{13}\ell_1'fr_1'r_{13} - c_1\ell_{13}\ell_3'hr_3'r_{13}) \\
&\quad - c_1(c_3\ell_{23}\ell_2''gr_2''r_{23} - c_2\ell_{23}\ell_3''hr_3''r_{23}) \\
&= c_2\ell_{13}(c_3\ell_1'fr_1' - c_1\ell_3'hr_3')r_{13} - c_1\ell_{23}(c_3\ell_2''gr_2'' - c_2\ell_3''hr_3'')r_{23}.
\end{aligned}$$

As placed in $\ell_1\mathrm{LM}(f)r_1 = \ell_3\mathrm{LM}(h)r_3$, if $\mathrm{LM}(f)$ and $\mathrm{LM}(h)$ overlap, then the S-polynomial corresponding to this overlap is[2] $\text{S-pol}(\ell_1', f, \ell_3', h)$. Similarly, if $\mathrm{LM}(g)$ and $\mathrm{LM}(h)$ overlap as placed in $\ell_2\mathrm{LM}(g)r_2 = \ell_3\mathrm{LM}(h)r_3$, then the S-polynomial corresponding to this overlap is $\text{S-pol}(\ell_2'', g, \ell_3'', h)$. By assumption, these S-polynomials reduce to zero using P, so there are expressions

$$c_3\ell_1'fr_1' - c_1\ell_3'hr_3' - \sum_{i=1}^{\alpha} u_ip_iv_i = 0 \tag{3.5}$$

[2]For completeness, we note that the S-polynomial corresponding to the overlap can also be of the form $\text{S-pol}(\ell_3', h, \ell_1', f)$; this (inconsequentially) swaps the first two terms of Equation (3.5).

and

$$c_3\ell_2''gr_2'' - c_2\ell_3''hr_3'' - \sum_{j=1}^{\beta} u_jp_jv_j = 0, \tag{3.6}$$

where the u_i, v_i, u_j and v_j are terms; and $p_i, p_j \in P$ for all i and j. Using Proposition 3.1.4, we can state that these expressions will still exist even if $\mathrm{LM}(f)$ and $\mathrm{LM}(h)$ do not overlap as placed in $\ell_1\mathrm{LM}(f)r_1 = \ell_3\mathrm{LM}(h)r_3$; and if $\mathrm{LM}(g)$ and $\mathrm{LM}(h)$ do not overlap as placed in $\ell_2\mathrm{LM}(g)r_2 = \ell_3\mathrm{LM}(h)r_3$. It follows that

$$\begin{aligned}
c_3(\text{S-pol}(\ell_1, f, \ell_2, g)) &= c_2\ell_{13}(c_3\ell_1'fr_1' - c_1\ell_3'hr_3')r_{13} - c_1\ell_{23}(c_3\ell_2''gr_2'' - c_2\ell_3''hr_3'')r_{23} \\
&= c_2\ell_{13}\left(\sum_{i=1}^{\alpha} u_ip_iv_i\right)r_{13} - c_1\ell_{23}\left(\sum_{j=1}^{\beta} u_jp_jv_j\right)r_{23} \\
&= \sum_{i=1}^{\alpha} c_2\ell_{13}u_ip_iv_ir_{13} - \sum_{j=1}^{\beta} c_1\ell_{23}u_jp_jv_jr_{23}; \\
\text{S-pol}(\ell_1, f, \ell_2, g) &= \sum_{i=1}^{\alpha} c_3^{-1}c_2\ell_{13}u_ip_iv_ir_{13} - \sum_{j=1}^{\beta} c_3^{-1}c_1\ell_{23}u_jp_jv_jr_{23}.
\end{aligned}$$

To conclude that the S-polynomial $\text{S-pol}(\ell_1, f, \ell_2, g)$ reduces to zero using P, it remains to show that the algebraic expression $-\sum_{i=1}^{\alpha} c_3^{-1}c_2\ell_{13}u_ip_iv_ir_{13} + \sum_{j=1}^{\beta} c_3^{-1}c_1\ell_{23}u_jp_jv_jr_{23}$ corresponds to a valid reduction of $\text{S-pol}(\ell_1, f, \ell_2, g)$. To do this, it is sufficient to show that no term in either of the summations is greater than the term $\ell_1\mathrm{LM}(f)r_1$ (so that $\mathrm{LM}(\ell_{13}u_ip_iv_ir_{13}) < \ell_1\mathrm{LM}(f)r_1$ and $\mathrm{LM}(\ell_{23}u_jp_jv_jr_{23}) < \ell_1\mathrm{LM}(f)r_1$ for all i and j). But this follows from Equation (3.4) and from the fact that the reductions of the expressions $c_3\ell_1'fr_1' - c_1\ell_3'hr_3'$ and $c_3\ell_2''gr_2'' - c_2\ell_3''hr_3''$ in Equations (3.5) and (3.6) are valid, so that $\mathrm{LM}(u_ip_iv_i) < \mathrm{LM}(\ell_1'fr_1')$ and $\mathrm{LM}(u_jp_jv_j) < \mathrm{LM}(\ell_2''gr_2'')$ for all i and j. $\square$

Remark 3.4.2 The three polynomials f, g and h in the above proposition do not necessarily have to be distinct (indeed, $f = g = h$ is allowed) — the only restriction is that the S-polynomial $\text{S-pol}(\ell_1, f, \ell_2, g)$ has to be different from the S-polynomials $\text{S-pol}(\ell_1', f, \ell_3', h)$ and $\text{S-pol}(\ell_2'', g, \ell_3'', h)$; for example, if $f = h$, then we cannot have $\ell_1' = \ell_3'$.

3.4.2 Homogeneous Gröbner Bases

Because it is computationally more expensive to do noncommutative polynomial arithmetic than it is to do commutative polynomial arithmetic, gains in efficiency due to working with homogeneous bases are even more significant in the noncommutative case.

For this reason, some systems for computing noncommutative Gröbner Bases will only work with homogeneous input bases, although (as in the commutative case) it is still sometimes possible to use these systems on non-homogeneous input bases by using the concepts of homogenisation, dehomogenisation and extendible monomial orderings.

Definition 3.4.3 Let $p = p_0 + \cdots + p_m$ be a polynomial over the polynomial ring $R\langle x_1, \ldots, x_n\rangle$, where each p_i is the sum of the degree i terms in p (we assume that $p_m \neq 0$). The *left homogenisation* of p with respect to a new (homogenising) variable y is the polynomial

$$h_\ell(p) := y^m p_0 + y^{m-1}p_1 + \cdots + yp_{m-1} + p_m;$$

and the *right homogenisation* of p with respect to a new (homogenising) variable y is the polynomial

$$h_r(p) := p_0 y^m + p_1 y^{m-1} + \cdots + p_{m-1}y + p_m.$$

Homogenised polynomials belong to polynomial rings determined by where y is placed in the lexicographical ordering of the variables.

Definition 3.4.4 The *dehomogenisation* of a polynomial p is the polynomial $d(p)$ given by substituting $y = 1$ in p, where y is the homogenising variable.

Definition 3.4.5 A monomial ordering O is *extendible* if, given any polynomial $p = t_1 + \cdots + t_\alpha$ ordered with respect to O (where $t_1 > \cdots > t_\alpha$), the homogenisation of p preserves the order on the terms ($t'_i > t'_{i+1}$ for all $1 \leqslant i \leqslant \alpha - 1$, where the homogenisation process maps the term $t_i \in p$ to the term t'_i).

In the noncommutative case, an extendible monomial ordering must specify how to homogenise a polynomial (by multiplying with the homogenising variable on the left or on the right) as well as stating where the new variable y appears in the ordering of the variables. Here are the conventions for those monomial orderings defined in Section 1.2.2 that are extendible, assuming that we start with a polynomial ring $R\langle x_1, \ldots, x_n\rangle$.

Monomial Ordering	Type of Homogenisation	Position of the new variable y in the ordering of the variables
InvLex	Right	$y < x_i$ for all x_i
DegLex	Left	$y < x_i$ for all x_i
DegInvLex	Left	$y > x_i$ for all x_i
DegRevLex	Right	$y > x_i$ for all x_i

Noncommutativity also provides the possibility of the new variable y becoming 'trapped' in the middle of some monomial forming part of a polynomial computed during the course of Mora's algorithm. For example, working with DegRevLex, consider the homogenised polynomial $h_r(x_1^2 + x_1) = x_1^2 + x_1y$ and the S-polynomial

$$\text{S-pol}(x_1, x_1^2 + x_1y, 1, x_1^2 + x_1y) = x_1(x_1^2 + x_1y) - (x_1^2 + x_1y)x_1 = x_1^2y - x_1yx_1.$$

Because y appears in the middle of the monomial x_1yx_1, the S-polynomial does not immediately reduce to zero as it does in the non-homogenised version of the S-polynomial,

$$\text{S-pol}(x_1, x_1^2 + x_1, 1, x_1^2 + x_1) = x_1(x_1^2 + x_1) - (x_1^2 + x_1)x_1 = 0.$$

We must therefore make certain that y only appears on one side of any given monomial by introducing the set of polynomials $H = \{h_1, h_2, \ldots, h_n\} = \{yx_1 - x_1y,\ yx_2 - x_2y,\ \ldots,\ yx_n - x_ny\}$ into our initial homogenised basis, ensuring that y commutes with all the other variables in the polynomial ring. This way, the first S-polynomial will reduce to zero as follows:

$$x_1^2y - x_1yx_1 \to_{h_1} x_1^2y - x_1^2y = 0.$$

Which side y will appear on will be determined by whether $\mathrm{LM}(yx_i - x_iy) = yx_i$ or $\mathrm{LM}(yx_i - x_iy) = x_iy$ in our chosen monomial ordering (pushing y to the right or to the left respectively). This side must match the method of homogenisation, which explains why Lex is not an extendible monomial ordering — for Lex to be extendible, we must homogenise on the right and have $y < x_i$ for all x_i, but then because $\mathrm{LM}(yx_i - x_iy) = x_iy$ with respect to Lex, the variable y will always in practice appear on the left.

Definition 3.4.6 Let $F = \{f_1, \ldots, f_m\}$ be a non-homogeneous set of polynomials over the polynomial ring $R\langle x_1, \ldots, x_n\rangle$. To compute a Gröbner Basis for F using a program that only accepts sets of homogeneous polynomials as input, we use the following procedure (which will only work in conjunction with an extendible monomial ordering).

(a) Construct a homogeneous set of polynomials $F' = \{h_\ell(f_1), \ldots, h_\ell(f_m)\}$ or $F' = \{h_r(f_1), \ldots, h_r(f_m)\}$ (dependent on the monomial ordering used).

(b) Compute a Gröbner Basis G' for the set $F' \cup H$, where $H = \{yx_1 - x_1y,\ yx_2 - x_2y,\ \ldots,\ yx_n - x_ny\}$.

(c) Dehomogenise each polynomial $g' \in G'$ to obtain a Gröbner Basis G for F, noting

that no polynomial originating from H will appear in G ($d(h_i) = 0$ for all $h_i \in H$).

3.4.3 Selection Strategies

As in the commutative case, the order in which S-polynomials are processed during Mora's algorithm has an important effect on the efficiency of the algorithm. Let us now generalise the selection strategies defined in Section 2.5.3 for use in the noncommutative setting, basing our decisions on the *overlap words* of S-polynomials.

Definition 3.4.7 The *overlap word* of an S-polynomial $\text{S-pol}(\ell_1, f, \ell_2, g) = \text{LC}(g)\ell_1 f r_1 - \text{LC}(f)\ell_2 g r_2$ is the monomial $\ell_1 \text{LM}(f) r_1$ ($= \ell_2 \text{LM}(g) r_2$).

Definition 3.4.8 In the noncommutative *normal strategy*, we choose an S-polynomial to process if its overlap word is minimal in the chosen monomial ordering amongst all such overlap words.

Definition 3.4.9 In the noncommutative *sugar strategy*, we choose an S-polynomial to process if its sugar (a value associated to the S-polynomial) is minimal amongst all such values (we use the normal strategy in the event of a tie).

The sugar of an S-polynomial is computed by using the following rules on the sugars of polynomials we encounter during the computation of a Gröbner Basis for the set of polynomials $F = \{f_1, \dots, f_m\}$.

(1) The sugar Sug_{f_i} of a polynomial $f_i \in F$ is the total degree of the polynomial f_i (which is the degree of the term of maximal degree in f_i).

(2) If p is a polynomial and if t_1 and t_2 are terms, then $\text{Sug}_{t_1 p t_2} = \deg(t_1) + \text{Sug}_p + \deg(t_2)$.

(3) If $p = p_1 + p_2$, then $\text{Sug}_p = \max(\text{Sug}_{p_1}, \text{Sug}_{p_2})$.

It follows that the sugar of the S-polynomial $\text{S-pol}(\ell_1, g, \ell_2, h) = \text{LC}(h)\ell_1 g r_1 - \text{LC}(g)\ell_2 h r_2$ is given by the formula

$$\text{Sug}_{\text{S-pol}(\ell_1, g, \ell_2, h)} = \max(\deg(\ell_1) + \text{Sug}_g + \deg(r_1),\ \deg(\ell_2) + \text{Sug}_h + \deg(r_2)).$$

3.4.4 Logged Gröbner Bases

Definition 3.4.10 Let $G = \{g_1, \dots, g_p\}$ be a noncommutative Gröbner Basis computed from an initial basis $F = \{f_1, \dots, f_m\}$. We say that G is a *Logged Gröbner Basis* if, for each $g_i \in G$, we have an explicit expression of the form

$$g_i = \sum_{\alpha=1}^{\beta} \ell_\alpha f_{k_\alpha} r_\alpha,$$

where the ℓ_α and the r_α are terms and $f_{k_\alpha} \in F$ for all $1 \leqslant \alpha \leqslant \beta$.

Proposition 3.4.11 *Let $F = \{f_1, \dots, f_m\}$ be a finite basis over a noncommutative polynomial ring. If we can compute a Gröbner Basis for F, then it is always possible to compute a Logged Gröbner Basis for F.*

Proof: We refer to the proof of Proposition 2.5.11, substituting

$$\text{S-pol}(\ell_1, f_i, \ell_2, f_j) - \sum_{\alpha=1}^{\beta} \ell_\alpha g_{k_\alpha} r_\alpha$$

for f_{m+1} (the ℓ_α and the r_α are terms). □

3.5 A Worked Example

To demonstrate Mora's algorithm in action, let us now calculate a Gröbner Basis for the ideal J generated by the set of polynomials $F := \{f_1, f_2, f_3\} = \{xy - z, yz + 2x + z, yz + x\}$ over the polynomial ring $\mathbb{Q}\langle x, y, z\rangle$. We shall use the DegLex monomial ordering (with $x > y > z$); use the normal selection strategy; calculate a Logged Gröbner Basis; and use Buchberger's criteria.

3.5.1 Initialisation

The first part of Mora's algorithm requires us to find all the overlaps between the lead monomials of the three polynomials in the initial basis $G := \{g_1, g_2, g_3\} = \{xy - z, yz + 2x + z, yz + x\}$. There are three overlaps in total, summarised by the following table.

	Overlap 1	Overlap 2	Overlap 3
Overlap Word	yz	xyz	xyz
Polynomial 1	$yz + 2x + z$	$xy - z$	$xy - z$
Polynomial 2	$yz + x$	$yz + 2x + z$	$yz + x$
ℓ_1	1	1	1
r_1	1	z	z
ℓ_2	1	x	x
r_2	1	1	1
Degree of Overlap Word	2	3	3

Because we are using the normal selection strategy, it is clear that Overlap 1 will appear in the list A first, but we are free to choose the order in which the other two overlaps appear (because their overlap words are identical). To eliminate this choice, we will use the following *tie-breaking strategy* to order any two S-polynomials whose overlap words are identical.

Definition 3.5.1 Let $s_1 = \text{S-pol}(\ell_1, g_a, \ell_2, g_b)$ and $s_2 = \text{S-pol}(\ell_3, g_c, \ell_4, g_d)$ be two S-polynomials with identical overlap words, where $g_a, g_b, g_c, g_d \in G = \{g_1, \dots, g_\alpha\}$. Assuming (without loss of generality) that $a < b$ and $c < d$, the *tie-breaking strategy* places s_1 before s_2 in A if $a < c$ or if $a = c$ and $b \leqslant d$; and later in A otherwise.

Applying the tie-breaking strategy for Overlaps 2 and 3, it follows that Overlap 2 = $\text{S-pol}(1, g_1, x, g_2)$ will appear in A before Overlap 3 = $\text{S-pol}(1, g_1, x, g_3)$.

Before we start the main part of the algorithm, let us note that for the Logged Gröbner Basis, we begin the algorithm with trivial expressions for each of the three polynomials in the initial basis G in terms of the polynomials of the input basis F: $g_1 = xy - z = f_1$; $g_2 = yz + 2x + z = f_2$; and $g_3 = yz + x = f_3$.

3.5.2 Calculating and Reducing S-polynomials

The first S-polynomial to analyse corresponds to Overlap 1 and is the polynomial

$$1(yz + 2x + z)1 - 1(yz + x)1 = 2x + z - x = x + z.$$

This polynomial is irreducible with respect to G, and so we add it to G to obtain a new basis $G = \{xy - z, yz + 2x + z, yz + x, x + z\} = \{g_1, g_2, g_3, g_4\}$. Looking for overlaps between

the lead monomial of $x+z$ and the lead monomials of the four elements of G, we see that there is one such overlap (with g_1) whose overlap word has degree 2, so this overlap is added to the beginning of the list A to obtain $A = \{$S-pol$(1, xy-z, 1, x+z)$, S-pol$(1, xy-z, x, yz+2x+z)$, S-pol$(1, xy-z, x, yz+x)\}$. As far as the Logged Gröbner Basis goes, $g_4 = x+z = 1(yz+2x+z)1 - 1(yz+x)1 = f_2 - f_3$.

The next entry in A produces the polynomial

$$1(xy-z)1 - 1(x+z)y = -zy - z.$$

As before, this polynomial is irreducible with respect to G, so we add it to G as the fifth element. There are also four overlaps between the lead monomial of $-zy-z$ and the lead monomials of the five polynomials in G:

	Overlap 1	Overlap 2	Overlap 3	Overlap 4
Overlap Word	zyz	zyz	yzy	yzy
Polynomial 1	$yz+2x+z$	$yz+x$	$yz+2x+z$	$yz+x$
Polynomial 2	$-zy-z$	$-zy-z$	$-zy-z$	$-zy-z$
ℓ_1	z	z	1	1
r_1	1	1	y	y
ℓ_2	1	1	y	y
r_2	z	z	1	1
Degree of Overlap Word	3	3	3	3

Inserting these overlaps into the list A, we obtain

$$\begin{aligned} A = \{ \quad & \text{S-pol}(z, yz+2x+z, 1, -zy-z),\ \text{S-pol}(z, yz+x, 1, -zy-z), \\ & \text{S-pol}(1, yz+2x+z, y, -zy-z),\ \text{S-pol}(1, yz+x, y, -zy-z), \\ & \text{S-pol}(1, xy-z, x, yz+2x+z),\ \text{S-pol}(1, xy-z, x, yz+x) \qquad \}. \end{aligned}$$

The logged representation of the fifth basis element again comes straight from the S-polynomial (as no reduction was performed), and is as follows: $g_5 = -zy-z = 1(xy-z)1 - 1(x+z)y = 1(f_1)1 - 1(f_2-f_3)y = f_1 - f_2y + f_3y$.

The next entry in A yields the polynomial

$$-z(yz+2x+z)1 - 1(-zy-z)z = -2zx - z^2 + z^2 = -2zx.$$

This time, the fourth polynomial in our basis reduces the S-polynomial in question, giving a reduction $-2zx \to_{g_4} 2z^2$. When we add this polynomial to G and add all five new overlaps to A, we are left with a six element basis $G = \{xy - z, yz + 2x + z, yz + x, x + z, -zy - z, 2z^2\}$ and a list

$$\begin{aligned}
A = \{ \quad & \text{S-pol}(1, 2z^2, z, 2z^2),\ \text{S-pol}(z, 2z^2, 1, 2z^2),\\
& \text{S-pol}(z, -zy - z, 1, 2z^2),\ \text{S-pol}(z, yz + x, 1, -zy - z),\\
& \text{S-pol}(1, yz + 2x + z, y, 2z^2),\ \text{S-pol}(1, yz + x, y, 2z^2),\\
& \text{S-pol}(1, yz + 2x + z, y, -zy - z),\ \text{S-pol}(1, yz + x, y, -zy - z),\\
& \text{S-pol}(1, xy - z, x, yz + 2x + z),\ \text{S-pol}(1, xy - z, x, yz + x) \qquad \}.
\end{aligned}$$

We obtain the logged version of the sixth basis element by working backwards through our calculations:

$$\begin{aligned}
g_6 &= 2z^2\\
&= -2zx + 2z(x + z)\\
&= (-z(yz + 2x + z)1 - 1(-zy - z)z) + 2z(x + z)\\
&= (-z(f_2) - (f_1 - f_2y + f_3y)z) + 2z(f_2 - f_3)\\
&= -f_1z + zf_2 + f_2yz - 2zf_3 - f_3yz.
\end{aligned}$$

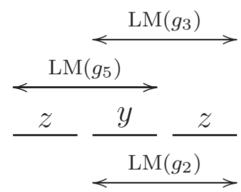
3.5.3 Applying Buchberger's Second Criterion

The next three entries in A all yield S-polynomials that are either zero or reduce to zero (for example, the first entry corresponds to the polynomial $2(2z^2)z - 2z(2z^2)1 = 4z^3 - 4z^3 = 0$). The fourth entry in A, S-pol$(z, yz + x, 1, -zy - z)$, then enables us (for the first time) to apply Buchberger's second criterion, allowing us to move on to look at the fifth entry of A. Before we do this however, let us explain why we can apply Buchberger's second criterion in this particular case.

Recall (from Proposition 3.4.1) that in order to apply Buchberger's second criterion for the S-polynomial S-pol$(z, yz + x, 1, -zy - z)$, we need to find a polynomial $g_i \in G$ such that LM(g_i) divides the overlap word of our S-polynomial, and any S-polynomials corresponding to overlaps (as positioned in the overlap word) between LM(g_i) and either LM$(yz + x)$ or LM$(-zy - z)$ reduce to zero using G (which will be the case if those

particular S-polynomials have been processed earlier in the algorithm).

Consider the polynomial $g_2 = yz + 2x + z$. The lead monomial of this polynomial divides the overlap word zyz of our S-polynomial, which we illustrate as follows.



As positioned in the overlap word, we note that $\mathrm{LM}(g_2)$ overlaps with both $\mathrm{LM}(g_3)$ and $\mathrm{LM}(g_5)$, with the overlaps corresponding to the S-polynomials $\text{S-pol}(1, g_2, 1, g_3) = \text{S-pol}(1, yz + 2x + z, 1, yz + x)$ and $\text{S-pol}(z, g_2, 1, g_5) = \text{S-pol}(z, yz + 2x + z, 1, -zy - z)$ respectively. But these S-polynomials have been processed earlier in the algorithm (they were the first and third S-polynomials to be processed); we can therefore apply Buchberger's second criterion in this instance.

There are now six S-polynomials left in A, all of whom either reduce to zero or are ignored due to Buchberger's second criterion. Here is a summary of what happens during the remainder of the algorithm.

S-polynomial	Action
$\text{S-pol}(1, yz + 2x + z, y, 2z^2)$	Reduces to zero using the division algorithm
$\text{S-pol}(1, yz + x, y, 2z^2)$	Ignored due to Buchberger's second criterion (using $yz + 2x + z$)
$\text{S-pol}(1, yz + 2x + z, y, -zy - z)$	Reduces to zero using the division algorithm
$\text{S-pol}(1, yz + x, y, -zy - z)$	Ignored due to Buchberger's second criterion (using $yz + 2x + z$)
$\text{S-pol}(1, xy - z, x, yz + 2x + z)$	Ignored due to Buchberger's second criterion (using $x + z$)
$\text{S-pol}(1, xy - z, x, yz + x)$	Ignored due to Buchberger's second criterion (using $yz + 2x + z$)

As the list A is now empty, the algorithm terminates with the following (Logged) Gröbner Basis.

Input Basis F	Gröbner Basis G
$f_1 = xy - z$	$g_1 = xy - z = f_1$
$f_2 = yz + 2x + z$	$g_2 = yz + 2x + z = f_2$
$f_3 = yz + x$	$g_3 = yz + x = f_3$
	$g_4 = x + z = f_2 - f_3$
	$g_5 = -zy - z = f_1 - f_2y + f_3y$
	$g_6 = 2z^2 = -f_1z + zf_2 + f_2yz - 2zf_3 - f_3yz$

3.5.4 Reduction

Now that we have constructed a Gröbner Basis for our ideal J, let us go on to find the unique reduced Gröbner Basis for J by applying Algorithm 6 to G.

In the first half of the algorithm, we must multiply each polynomial by the inverse of its lead coefficient and remove from the basis each polynomial whose lead monomial is a multiple of the lead monomial of some other polynomial in the basis. For the Gröbner Basis in question, we multiply g_5 by -1 and g_6 by $\frac{1}{2}$; and we remove g_1 and g_2 from the basis (because $\mathrm{LM}(g_1) = \mathrm{LM}(g_4) \times y$ and $\mathrm{LM}(g_2) = \mathrm{LM}(g_3)$). This leaves us with the following (minimal) Gröbner Basis.

Input Basis F	Gröbner Basis G
$f_1 = xy - z$	$g_3 = yz + x = f_3$
$f_2 = yz + 2x + z$	$g_4 = x + z = f_2 - f_3$
$f_3 = yz + x$	$g_5 = zy + z = -f_1 + f_2y - f_3y$
	$g_6 = z^2 = -\frac{1}{2}f_1z + \frac{1}{2}zf_2 + \frac{1}{2}f_2yz - zf_3 - \frac{1}{2}f_3yz$

In the second half of the algorithm, we reduce each $g_i \in G$ with respect to $(G \setminus \{g_i\}) \cup G'$, placing the remainder in the (initially empty) set G' and removing g_i from G. For the Gröbner Basis in question, we summarise what happens in the following table, noting that the only reduction that takes place is the reduction $yz + x \to_{g_4} yz + x - (x + z) = yz - z$.

G	G'	g_i	g'_i
$\{yz+x, x+z, zy+z, z^2\}$	$\emptyset$	$yz+x$	$yz-z$
$\{x+z, zy+z, z^2\}$	$\{yz-z\}$	$x+z$	$x+z$
$\{zy+z, z^2\}$	$\{yz-z, x+z\}$	$zy+z$	$zy+z$
$\{z^2\}$	$\{yz-z, x+z, zy+z\}$	z^2	z^2
$\emptyset$	$\{yz-z, x+z, zy+z, z^2\}$		

We can now give the unique reduced (Logged) Gröbner Basis for J.

Input Basis F	Unique Reduced Gröbner Basis G'
$f_1 = xy - z$	$yz - z = -f_2 + 2f_3$
$f_2 = yz + 2x + z$	$x + z = f_2 - f_3$
$f_3 = yz + x$	$zy + z = -f_1 + f_2y - f_3y$
	$z^2 = -\frac{1}{2}f_1z + \frac{1}{2}zf_2 + \frac{1}{2}f_2yz - zf_3 - \frac{1}{2}f_3yz$

Chapter 4

Commutative Involutive Bases

Given a Gröbner Basis G for an ideal J over a polynomial ring $\mathcal{R}$, we know that the remainder of any polynomial $p \in \mathcal{R}$ with respect to G is unique. But although this remainder is unique, there may be many ways of obtaining the remainder, as it is possible that several polynomials in G divide our polynomial p, giving several *reduction paths* for p.

Example 4.0.2 Consider the DegLex Gröbner Basis $G := \{g_1, g_2, g_3\} = \{x^2 - 2xy + 3,\ 2xy + y^2 + 5,\ \frac{5}{4}y^3 - \frac{5}{2}x + \frac{37}{4}y\}$ over the polynomial ring $\mathcal{R} := \mathbb{Q}[x, y]$ from Example 2.3.2, and consider the polynomial $p := x^2y + y^3 + 8y \in \mathcal{R}$. The remainder of p with respect to G is 0 (so that p is a member of the ideal J generated by G), but there are two ways of obtaining this remainder, as shown in the following diagram.

$$\begin{tikzcd}
& x^2y + y^3 + 8y \arrow[dl, "g_1"'] \arrow[dr, "g_2"] & \\
2xy^2 + y^3 + 5y \arrow[d, "g_2"'] & & -\frac{1}{2}xy^2 + y^3 - \frac{5}{2}x + 8y \arrow[d, "g_2"] \\
0 & & \frac{5}{4}y^3 - \frac{5}{2}x + \frac{37}{4}y \arrow[d, "g_3"] \\
& & 0
\end{tikzcd} \tag{4.1}$$

An Involutive Basis is a Gröbner Basis G for J such that there is only one possible reduction path for any polynomial $p \in \mathcal{R}$. In order to find such a basis, we must restrict

which reductions or divisions may take place by requiring, for each potential reduction of a polynomial p by a polynomial $g_i \in G$ (so that $\mathrm{LM}(p) = \mathrm{LM}(g_i) \times u$ for some monomial u), some extra conditions on the variables in u to be satisfied, namely that all variables in u have to be in a set of *multiplicative variables* for g_i, a set that is determined by a particular choice of an *involutive division*.

4.1 Involutive Divisions

In Definition 1.2.9, we saw that a commutative monomial u_1 is divisible by another monomial u_2 if there exists a third monomial u_3 such that $u_1 = u_2u_3$; we also introduced the notation $u_2 \mid u_1$ to denote that u_2 is a divisor of u_1, a divisor we shall now refer to as a *conventional* divisor of u_1. For a particular choice of an involutive division I, we say that u_2 is an *involutive* divisor of u_1, written $u_2 \mid_I u_1$, if, given a partitioning (by I) of the variables in the polynomial ring into sets of *multiplicative* and *nonmultiplicative* variables for u_2, all variables in u_3 are in the set of multiplicative variables for u_2.

Example 4.1.1 Let $u_1 := xy^2z^2$; $u_1' := x^2yz$ and $u_2 := xz$ be three monomials over the polynomial ring $\mathcal{R} := \mathbb{Q}[x, y, z]$, and let an involutive division I partition the variables in $\mathcal{R}$ into the following two sets of variables for the monomial u_2: multiplicative $= \{y, z\}$; nonmultiplicative $= \{x\}$. It is true that u_2 conventionally divides both monomials u_1 and u_1', but u_2 only involutively divides monomial u_1 as, defining $u_3 := y^2z$ and $u_3' := xy$ (so that $u_1 = u_2u_3$ and $u_1' = u_2u_3'$), we observe that all variables in u_3 are in the set of multiplicative variables for u_2, but the variables in u_3' (in particular the variable x) are not all in the set of multiplicative variables for u_2.

More formally, an involutive division I works with a set of monomials U over a polynomial ring $R[x_1, \ldots, x_n]$ and assigns a set of multiplicative variables $\mathcal{M}_I(u, U) \subseteq \{x_1, \ldots, x_n\}$ to each element $u \in U$. It follows that, *with respect to* U, a monomial w is divisible by a monomial $u \in U$ if $w = uv$ for some monomial v and all the variables that appear in v also appear in the set $\mathcal{M}_I(u, U)$.

Definition 4.1.2 Let M denote the set of all monomials in the polynomial ring $\mathcal{R} = R[x_1, \ldots, x_n]$, and let $U \subset M$. The *involutive cone* $\mathcal{C}_I(u, U)$ of any monomial $u \in U$ with respect to some involutive division I is defined as follows.

$$\mathcal{C}_I(u, U) = \{uv \text{ such that } v \in M \text{ and } u \mid_I uv\}.$$

Remark 4.1.3 We may think of an involutive cone of a particular monomial u as containing all monomials that are involutively divisible by u.

Up to now, we have not mentioned any restriction on how we may assign multiplicative variables to a particular set of monomials. Let us now specify the rules that ensure that a particular scheme of assigning multiplicative variables may be referred to as an involutive division.

Definition 4.1.4 Let M denote the set of all monomials in the polynomial ring $\mathcal{R} = R[x_1, \ldots, x_n]$. An *involutive division* I on M is defined if, given any finite set of monomials $U \subset M$, we can assign a set of *multiplicative variables* $\mathcal{M}_I(u, U) \subseteq \{x_1, \ldots, x_n\}$ to any monomial $u \in U$ such that the following two conditions are satisfied.

(a) If there exist two monomials $u_1, u_2 \in U$ such that $\mathcal{C}_I(u_1, U) \cap \mathcal{C}_I(u_2, U) \neq \emptyset$, then either $\mathcal{C}_I(u_1, U) \subset \mathcal{C}_I(u_2, U)$ or $\mathcal{C}_I(u_2, U) \subset \mathcal{C}_I(u_1, U)$.

(b) If $V \subset U$, then $\mathcal{M}_I(v, U) \subseteq \mathcal{M}_I(v, V)$ for all $v \in V$.

Remark 4.1.5 Informally, condition (a) above ensures that a monomial can only appear in two involutive cones $\mathcal{C}_I(u_1, U)$ and $\mathcal{C}_I(u_2, U)$ if u_1 is an involutive divisor of u_2 or vice-versa; while condition (b) ensures that the multiplicative variables of a polynomial $v \in V \subset U$ with respect to U all appear in the set of multiplicative variables of v with respect to V.

Definition 4.1.6 Given an involutive division I, the involutive span $\mathcal{C}_I(U)$ of a set of monomials U with respect to I is given by the expression

$$\mathcal{C}_I(U) = \bigcup_{u \in U} \mathcal{C}_I(u, U).$$

Remark 4.1.7 The (conventional) span of a set of monomials U is given by the expression

$$\mathcal{C}(U) = \bigcup_{u \in U} \mathcal{C}(u, U),$$

where $\mathcal{C}(u, U) = \{uv \mid v \text{ is a monomial}\}$ is the (conventional) cone of a monomial $u \in U$.

Definition 4.1.8 If an involutive division I determines the multiplicative variables for a monomial $u \in U$ independent of the set U, then I is a *global* division. Otherwise, I is a *local* division.

Remark 4.1.9 The multiplicative variables for a set of polynomials P (whose terms are ordered by a monomial ordering O) are determined by the multiplicative variables for the set of leading monomials $\mathrm{LM}(P)$.

4.1.1 Involutive Reduction

In Algorithm 7, we specify how to involutively divide a polynomial p with respect to a set of polynomials P.

Algorithm 7 The Commutative Involutive Division Algorithm

Input: A nonzero polynomial p and a set of nonzero polynomials $P = \{p_1, \dots, p_m\}$ over a polynomial ring $R[x_1, \dots x_n]$; an admissible monomial ordering O; an involutive division I.

Output: $\mathrm{Rem}_I(p, P) := r$, the involutive remainder of p with respect to P.

$r = 0$;
while $(p \neq 0)$ **do**
  $u = \mathrm{LM}(p)$; $c = \mathrm{LC}(p)$; $j = 1$; found = false;
  while $(j \leqslant m)$ **and** (found == false) **do**
    if $(\mathrm{LM}(p_j) \mid_I u)$ **then**
      found = true; $u' = u/\mathrm{LM}(p_j)$; $p = p - (c\mathrm{LC}(p_j)^{-1})p_j u'$;
    else
      $j = j + 1$;
    end if
  end while
  if (found == false) **then**
    $r = r + \mathrm{LT}(p)$; $p = p - \mathrm{LT}(p)$;
  end if
end while
return r;

Remark 4.1.10 The only difference between Algorithms 1 and 7 is that the line "**if** $(\mathrm{LM}(p_j) \mid u)$ **then**" in Algorithm 1 has been changed to the line "**if** $(\mathrm{LM}(p_j) \mid_I u)$ **then**" in Algorithm 7.

Definition 4.1.11 If the polynomial r is obtained by involutively dividing (with respect to some involutive division I) the polynomial p by one of (a) a polynomial q; (b) a sequence

of polynomials $q_1, q_2, \ldots, q_\alpha$; or (c) a set of polynomials Q, we will use the notation $p \xrightarrow[I]{}_q r$; $p \xrightarrow[I]{*} r$ and $p \xrightarrow[I]{}_Q r$ respectively (matching the notation introduced in Definition 1.2.16).

4.1.2 Thomas, Pommaret and Janet divisions

Let us now consider three different involutive divisions, all named after their creators in the theory of Partial Differential Equations (see [52], [47] and [35]).

Definition 4.1.12 (Thomas) Let $U = \{u_1, \ldots, u_m\}$ be a set of monomials over a polynomial ring $R[x_1, \ldots, x_n]$, where the monomial $u_j \in U$ (for $1 \leqslant j \leqslant m$) has corresponding multidegree $(e_j^1, e_j^2, \ldots, e_j^n)$. The *Thomas* involutive division $\mathcal{T}$ assigns multiplicative variables to elements of U as follows: the variable x_i is multiplicative for monomial u_j (written $x_i \in \mathcal{M}_{\mathcal{T}}(u_j, U)$) if $e_j^i = \max_k e_k^i$ for all $1 \leqslant k \leqslant m$.

Definition 4.1.13 (Pommaret) Let u be a monomial over a polynomial ring $R[x_1, \ldots, x_n]$ with multidegree $(e^1, e^2, \ldots, e^n)$. The *Pommaret* involutive division $\mathcal{P}$ assigns multiplicative variables to u as follows: if $1 \leqslant i \leqslant n$ is the smallest integer such that $e^i > 0$, then all variables $x_1, x_2, \ldots, x_i$ are multiplicative for u (we have $x_j \in \mathcal{M}_{\mathcal{P}}(u)$ for all $1 \leqslant j \leqslant i$).

Definition 4.1.14 (Janet) Let $U = \{u_1, \ldots, u_m\}$ be a set of monomials over a polynomial ring $R[x_1, \ldots, x_n]$, where the monomial $u_j \in U$ (for $1 \leqslant j \leqslant m$) has corresponding multidegree $(e_j^1, e_j^2, \ldots, e_j^n)$. The *Janet* involutive division $\mathcal{J}$ assigns multiplicative variables to elements of U as follows: the variable x_n is multiplicative for monomial u_j (written $x_n \in \mathcal{M}_{\mathcal{J}}(u_j, U)$) if $e_j^n = \max_k e_k^n$ for all $1 \leqslant k \leqslant m$; the variable x_i (for $1 \leqslant i < n$) is multiplicative for monomial u_j (written $x_i \in \mathcal{M}_{\mathcal{J}}(u_j, U)$) if $e_j^i = \max_k e_k^i$ for all monomials $u_k \in U$ such that $e_j^l = e_k^l$ for all $i < l \leqslant n$.

Remark 4.1.15 Thomas and Janet are local involutive divisions; Pommaret is a global involutive division.

Example 4.1.16 Let $U := \{x^5y^2z,\ x^4yz^2,\ x^2y^2z,\ xyz^3,\ xz^3,\ y^2z,\ z\}$ be a set of monomials over the polynomial ring $\mathbb{Q}[x, y, z]$, with $x > y > z$. Here are the multiplicative variables for U according to the three involutive divisions defined above.

Monomial	Thomas	Pommaret	Janet
x^5y^2z	$\{x,y\}$	$\{x\}$	$\{x,y\}$
x^4yz^2	$\emptyset$	$\{x\}$	$\{x,y\}$
x^2y^2z	$\{y\}$	$\{x\}$	$\{y\}$
xyz^3	$\{z\}$	$\{x\}$	$\{x,y,z\}$
xz^3	$\{z\}$	$\{x\}$	$\{x,z\}$
y^2z	$\{y\}$	$\{x,y\}$	$\{y\}$
z	$\emptyset$	$\{x,y,z\}$	$\{x\}$

Proposition 4.1.17 *All three involutive divisions defined above satisfy the conditions of Definition 4.1.4.*

Proof: Throughout, let M denote the set of all monomials in the polynomial ring $\mathcal{R} = R[x_1, \ldots, x_n]$; let $U = \{u_1, \ldots, u_m\} \subset M$ be a set of monomials with corresponding multidegrees $(e_k^1, e_k^2, \ldots, e_k^n)$ (where $1 \leqslant k \leqslant m$); let $u_i, u_j \in U$ (where $1 \leqslant i, j \leqslant m$, $i \neq j$); and let $m_1, m_2 \in M$ be two monomials with corresponding multidegrees $(f_1^1, f_1^2, \ldots, f_1^n)$ and $(f_2^1, f_2^2, \ldots, f_2^n)$. For condition (a), we need to show that if there exists a monomial $m \in M$ such that $m_1u_i = m = m_2u_j$ and all variables in m_1 and m_2 are multiplicative for u_i and u_j respectively, then either u_i is an involutive divisor of u_j or vice-versa. For condition (b), we need to show that all variables that are multiplicative for $u_i \in U$ are still multiplicative for $u_i \in V \subseteq U$.

Thomas. (a) It is sufficient to prove that $u_i = u_j$. Assume to the contrary that $u_i \neq u_j$, so that there is some $1 \leqslant k \leqslant n$ such that $e_i^k \neq e_j^k$. Without loss of generality, assume that $e_i^k < e_j^k$. Because $e_i^k + f_1^k = e_j^k + f_2^k$, it follows that $f_1^k > 0$ so that the variable x_k must be multiplicative for the monomial u_i. But this contradicts the fact that x_k cannot be multiplicative for u_i in the Thomas involutive division because $e_j^k > e_i^k$. We therefore have $u_i = u_j$.

(b) By definition, if $x_j \in \mathcal{M}_\mathcal{T}(u_i, U)$, then $e_i^j = \max_k e_k^j$ for all $u_k \in U$. Given a set $V \subseteq U$, it is clear that $e_i^j = \max_k e_k^j$ for all $u_k \in V$, so that $x_j \in \mathcal{M}_\mathcal{T}(u_i, V)$ as required.

Pommaret. (a) Let α and β ($1 \leqslant \alpha, \beta \leqslant n$) be the smallest integers such that $e_i^\alpha > 0$ and $e_j^\beta > 0$ respectively, and assume (without loss of generality) that $\alpha \geqslant \beta$. By definition, we must have $f_1^k = f_2^k = 0$ for all $\alpha < k \leqslant n$ because the x_k are all nonmultiplicative for u_i and u_j. It follows that $e_i^k = e_j^k$ for all $\alpha < k \leqslant n$. If $\alpha = \beta$, then it is clear that u_i is an involutive divisor of u_j if $e_i^\alpha < e_j^\alpha$, and u_j is an involutive divisor of u_i if $e_i^\alpha > e_j^\alpha$. If

$\alpha > \beta$, then $f_2^\alpha = 0$ as variable x_α is nonmultiplicative for u_j, so it follows that $e_i^\alpha \leqslant e_j^\alpha$ and hence u_i is an involutive divisor of u_j.

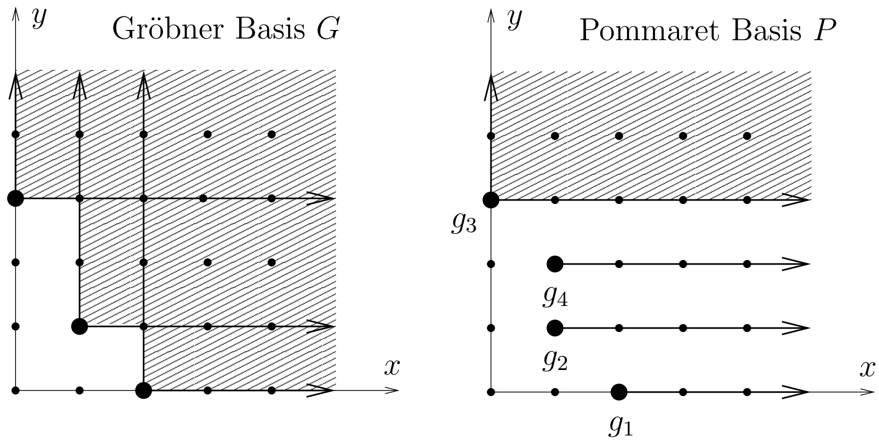
(b) Follows immediately because Pommaret is a global involutive division.

Janet. (a) We prove that $u_i = u_j$. Assume to the contrary that $u_i \neq u_j$, so there exists a maximal $1 \leqslant k \leqslant n$ such that $e_i^k \neq e_j^k$. Without loss of generality, assume that $e_i^k < e_j^k$. If $k = n$, we get an immediate contradiction because Janet is equivalent to Thomas for the final variable. If $k = n-1$, then because $e_i^{n-1} + f_1^{n-1} = e_j^{n-1} + f_2^{n-1}$, it follows that $f_1^{n-1} > 0$ so that the variable x_{n-1} must be multiplicative for the monomial u_i. But this contradicts the fact that x_{n-1} cannot be multiplicative for u_i in the Janet involutive division because $e_j^{n-1} > e_i^{n-1}$ and $e_j^n = e_i^n$. By induction on k, we can show that $e_i^k = e_j^k$ for all $1 \leqslant k \leqslant n$, so that $u_i = u_j$ as required.

(b) By definition, if $x_j \in \mathcal{M}_\mathcal{J}(u_i, U)$, then $e_i^j = \max_k e_k^j$ for all monomials $u_k \in U$ such that $e_i^l = e_k^l$ for all $i < l \leqslant n$. Given a set $V \subseteq U$, it is clear that $e_i^j = \max_k e_k^j$ for all $u_k \in V$ such that $e_i^l = e_k^l$ for all $i < l \leqslant n$, so that $x_j \in \mathcal{M}_\mathcal{J}(u_i, V)$ as required. $\square$

The conditions of Definition 4.1.4 ensure that any polynomial is involutively divisible by at most one polynomial in any Involutive Basis. One advantage of this important combinatorial property is that the Hilbert function of an ideal J is easily computable with respect to an Involutive Basis (see [4]).

Example 4.1.18 Returning to Example 4.0.2, consider again the DegLex Gröbner Basis $G := \{x^2 - 2xy + 3, 2xy + y^2 + 5, \frac{5}{4}y^3 - \frac{5}{2}x + \frac{37}{4}y\}$ over the polynomial ring $\mathbb{Q}[x, y]$. A Pommaret Involutive Basis for G is the set $P := G \cup \{g_4 := -5xy^2 - 5x + 6y\}$, with the variable x being multiplicative for all polynomials in P, and the variable y being multiplicative for just g_3. We can illustrate the difference between the overlapping cones of G and the non-overlapping involutive cones of P by the following diagram.



The diagram also demonstrates that the polynomial $p := x^2y + y^3 + 8y$ is initially conventionally divisible by two members of the Gröbner Basis G (as seen in Equation (4.1)), but is only involutively divisible by one member of the Involutive Basis P, starting the following unique involutive reduction path for p.

$$\begin{array}{c} x^2y + y^3 + 8y \\ \big\downarrow{\scriptstyle g_2} \\ -\frac{1}{2}xy^2 + y^3 - \frac{5}{2}x + 8y \\ \big\downarrow{\scriptstyle g_4} \\ y^3 - 2x + \frac{37}{5}y \\ \big\downarrow{\scriptstyle g_3} \\ 0 \end{array}$$

4.2 Prolongations and Autoreduction

Whereas Buchberger's algorithm constructs a Gröbner Basis by using S-polynomials, the involutive algorithm will construct an Involutive Basis by using *prolongations* and *autoreduction*.

Definition 4.2.1 Given a set of polynomials P, a *prolongation* of a polynomial $p \in P$ is a product px_i, where $x_i \notin \mathcal{M}_I(\mathrm{LM}(p), \mathrm{LM}(P))$ with respect to some involutive division I.

Definition 4.2.2 A set of polynomials P is said to be *autoreduced* if no polynomial $p \in P$ exists such that p contains a term which is involutively divisible (with respect to P) by some polynomial $p' \in P \setminus \{p\}$. Algorithm 8 provides a way of performing autoreduction,

and introduces the following notation: Let $\mathrm{Rem}_I(A, B, C)$ denote the involutive remainder of the polynomial A with respect to the set of polynomials B, where reductions are only to be performed by elements of the set $C \subseteq B$.

Remark 4.2.3 The involutive cones associated to an autoreduced set of polynomials are always disjoint, meaning that a given monomial can only appear in at most one of the involutive cones.

Algorithm 8 The Commutative Autoreduction Algorithm

Input: A set of polynomials $P = \{p_1, p_2, \ldots, p_\alpha\}$; an involutive division I.
Output: An autoreduced set of polynomials $Q = \{q_1, q_2, \ldots, q_\beta\}$.

```
while (∃ p_i ∈ P such that Rem_I(p_i, P, P \ {p_i}) ≠ p_i) do
  p'_i = Rem_I(p_i, P, P \ {p_i});
  P = P \ {p_i};
  if (p'_i ≠ 0) then
    P = P ∪ {p'_i};
  end if
end while
Q = P;
return Q;
```

Proposition 4.2.4 *Let P be a set of polynomials over a polynomial ring $\mathcal{R} = R[x_1, \ldots, x_n]$, and let f and g be two polynomials also in $\mathcal{R}$. If P is autoreduced with respect to an involutive division I, then* $\mathrm{Rem}_I(f, P) + \mathrm{Rem}_I(g, P) = \mathrm{Rem}_I(f + g, P)$.

Proof: Let $f' := \mathrm{Rem}_I(f, P)$; $g' := \mathrm{Rem}_I(g, P)$ and $h' := \mathrm{Rem}_I(h, P)$, where $h := f + g$. Then, by the respective involutive reductions, we have expressions

$$f' = f - \sum_{a=1}^{A} p_{\alpha_a} t_a;$$

$$g' = g - \sum_{b=1}^{B} p_{\beta_b} t_b$$

and

$$h' = h - \sum_{c=1}^{C} p_{\gamma_c} t_c,$$

where p_{α_a}, p_{β_b}, $p_{\gamma_c} \in P$ and t_a, t_b, t_c are terms which are multiplicative (over P) for each p_{α_a}, p_{β_b} and p_{γ_c} respectively.

Consider the polynomial $h' - f' - g'$. By the above expressions, we can deduce[1] that

$$h' - f' - g' = \sum_{a=1}^{A} p_{\alpha_a} t_a + \sum_{b=1}^{B} p_{\beta_b} t_b - \sum_{c=1}^{C} p_{\gamma_c} t_c =: \sum_{d=1}^{D} p_{\delta_d} t_d.$$

Claim: $\mathrm{Rem}_I(h' - f' - g', P) = 0$.

Proof of Claim: Let t denote the leading term of the polynomial $\sum_{d=1}^{D} p_{\delta_d} t_d$. Then $\mathrm{LM}(t) = \mathrm{LM}(p_{\delta_k} t_k)$ for some $1 \leqslant k \leqslant D$ since, if not, there exists a monomial $\mathrm{LM}(p_{\delta_{k'}} t_{k'}) = \mathrm{LM}(p_{\delta_{k''}} t_{k''}) =: u$ for some $1 \leqslant k', k'' \leqslant D$ (with $p_{\delta_{k'}} \neq p_{\delta_{k''}}$) such that u is involutively divisible by the two polynomials $p_{\delta_{k'}}$ and $p_{\delta_{k''}}$, contradicting Definition 4.1.4 (recall that our set P is autoreduced, so that the involutive cones of P are disjoint). It follows that we can use p_{δ_k} to eliminate t by involutively reducing $h' - f' - g'$ as shown below.

$$\sum_{d=1}^{D} p_{\delta_d} t_d \xrightarrow[I]{} {}_{p_{\delta_k}} \sum_{d=1}^{k-1} p_{\delta_d} t_d + \sum_{d=k+1}^{D} p_{\delta_d} t_d. \tag{4.2}$$

By induction, we can apply a chain of involutive reductions to the right hand side of Equation (4.2) to obtain a zero remainder, so that $\mathrm{Rem}_I(h' - f' - g', P) = 0$. □

To complete the proof, we note that since f', g' and h' are all involutively irreducible, we must have $\mathrm{Rem}_I(h' - f' - g', P) = h' - f' - g'$. It therefore follows that $h' - f' - g' = 0$, or $h' = f' + g'$ as required. □

Remark 4.2.5 The above proof is based on the proofs of Theorem 5.4 and Corollary 5.5 in [25].

Let us now give a definition of a Locally Involutive Basis in terms of prolongations. Later on in this chapter, we will discover that the Involutive Basis algorithm only constructs Locally Involutive Bases, and it is the extra properties of each involutive division used with the algorithm that ensures that any computed Locally Involutive Basis is an Involutive Basis.

Definition 4.2.6 Given an involutive division I and an admissible monomial ordering

[1] For $1 \leqslant d \leqslant A$, $p_{\delta_d} t_d = p_{\alpha_a} t_a$ $(1 \leqslant a \leqslant A)$; for $A+1 \leqslant d \leqslant A+B$, $p_{\delta_d} t_d = p_{\beta_b} t_b$ $(1 \leqslant b \leqslant B)$; and for $A+B+1 \leqslant d \leqslant A+B+C =: D$, $p_{\delta_d} t_d = p_{\gamma_c} t_c$ $(1 \leqslant c \leqslant C)$.

O, an autoreduced set of polynomials P is a *Locally Involutive Basis* with respect to I and O if any prolongation of any polynomial $p_i \in P$ involutively reduces to zero using P.

Definition 4.2.7 Given an involutive division I and an admissible monomial ordering O, an autoreduced set of polynomials P is an *Involutive Basis* with respect to I and O if any multiple $p_i t$ of any polynomial $p_i \in P$ by any term t involutively reduces to zero using P.

4.3 Continuity and Constructivity

In the theory of commutative Gröbner Bases, Buchberger's algorithm returns a Gröbner Basis as long as an admissible monomial ordering is used. In the theory of commutative Involutive Bases however, not only must an admissible monomial ordering be used, but the involutive division chosen must be *continuous* and *constructive*.

Definition 4.3.1 (Continuity) Let I be an involutive division, and let U be an arbitrary set of monomials over a polynomial ring $R[x_1, \dots, x_n]$. We say that I is *continuous* if, given any sequence of monomials $\{u_1, u_2, \dots, u_m\}$ from U such that for all $i < m$, we have $u_{i+1} \mid_I u_i x_{j_i}$ for some variable x_{j_i} that is nonmultiplicative for monomial m_i (or $x_{j_i} \notin \mathcal{M}_I(u_i, U)$), no two monomials in the sequence are the same ($u_r \neq u_s$ for all $r \neq s$, where $1 \leqslant r, s \leqslant m$).

Proposition 4.3.2 *The Thomas, Pommaret and Janet involutive divisions are all continuous.*

Proof: Throughout, let the sequence of monomials $\{u_1, \dots, u_i, \dots, u_m\}$ have corresponding multidegrees $(e_i^1, e_i^2, \dots, e_i^n)$ (where $1 \leqslant i \leqslant m$).

Thomas. If the variable x_{j_i} is nonmultiplicative for monomial u_i, then, by definition, $e_i^{j_i} \neq \max_t e_t^{j_i}$ for all $u_t \in U$. Variable x_{j_i} cannot therefore be multiplicative for monomial u_{i+1} if $e_{i+1}^{j_i} \leqslant e_i^{j_i}$, so we must have $e_{i+1}^{j_i} = e_i^{j_i} + 1$ in order to have $u_{i+1} \mid_{\mathcal{T}} u_i x_{j_i}$. Further, for all $1 \leqslant k \leqslant n$ such that $k \neq j_i$, we must have $e_{i+1}^k = e_i^k$ as, if $e_{i+1}^k < e_i^k$, then x_k cannot be multiplicative for monomial u_{i+1} (which contradicts $u_{i+1} \mid_{\mathcal{T}} u_i x_{j_i}$). Thus $u_{i+1} = u_i x_{j_i}$, and so it is clear that the monomials in the sequence $\{u_1, u_2, \dots, u_m\}$ are all different.

Pommaret. Let α_i ($1 \leqslant \alpha_i \leqslant n$) be the smallest integer such that $e_i^{\alpha_i} > 0$ (where $1 \leqslant i \leqslant m$), so that $e_i^k = 0$ for all $k < \alpha_i$. Because $u_{i+1} \mid_{\mathcal{P}} u_i x_{j_i}$ for all $1 \leqslant i < m$,

and because (by definition) $j_i > \alpha_i$, it follows that we must have $e_{i+1}^k = 0$ for all $k < \alpha_i$. Therefore $\alpha_{i+1} \geqslant \alpha_i$ for all $1 \leqslant i < n$. If $\alpha_{i+1} = \alpha_i$, we note that $e_{i+1}^{\alpha_i} \leqslant e_i^{\alpha_i}$ because variable x_{α_i} is multiplicative for monomial u_{i+1}. If then we have $e_{i+1}^{\alpha_i} = e_i^{\alpha_i}$, then because the variable x_{j_i} is also nonmultiplicative for monomial u_{i+1}, we must have $e_{i+1}^{j_i} = e_i^{j_i} + 1$.

It is now clear that the monomials in the sequence $\{u_1, u_2, \ldots, u_m\}$ are all different because (a) the values in the sequence $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ monotonically increase; (b) for consecutive values $\alpha_s, \alpha_{s+1}, \ldots, \alpha_{s+\sigma}$ in α that are identical ($1 \leqslant s < m$, $s + \sigma \leqslant m$), the values in the corresponding sequence $E = \{e_s^{\alpha_s}, e_{s+1}^{\alpha_s}, \ldots, e_{s+\sigma}^{\alpha_s}\}$ monotonically decrease; (c) for consecutive values $e_t^{\alpha_s}, e_{t+1}^{\alpha_s}, \ldots, e_{t+\tau}^{\alpha_s}$ in E that are identical ($s \leqslant t < s+\sigma$, $t+\tau \leqslant s+\sigma$), the degrees of the monomials $u_t, u_{t+1}, \ldots, u_{t+\tau}$ strictly increase.

Janet. Consider the monomials u_1, u_2 and the variable x_{j_1} that is nonmultiplicative for u_1. We will first prove (by induction) that $e_2^i = e_1^i$ for all $j_1 < i \leqslant n$. For the case $i = n$, we must have $e_2^n = e_1^n$ otherwise (by definition) variable x_n is nonmultiplicative for monomial u_2 (we have $e_2^n < e_1^n$), contradicting that fact that $u_2 \mid_{\mathcal{J}} u_1 x_{j_1}$. For the inductive step, assume that $e_2^i = e_1^i$ for all $k \leqslant i \leqslant n$, and let us look at the case $i = k-1$. If $e_2^{k-1} < e_1^{k-1}$, then (by definition) variable x_{k-1} is nonmultiplicative for monomial u_2, again contradicting the fact that $u_2 \mid_{\mathcal{J}} u_1 x_{j_1}$. It follows that we must have $e_2^{k-1} = e_1^{k-1}$.

Let us now prove that $e_2^{j_1} = e_1^{j_1} + 1$. We can rule out the case $e_2^{j_1} < e_1^{j_1}$ immediately because this implies that the variable x_{j_1} is nonmultiplicative for monomial u_2 (by definition), contradicting the fact that $u_2 \mid_{\mathcal{J}} u_1 x_{j_1}$. The case $e_2^{j_1} = e_1^{j_1}$ can also be ruled out because we cannot have $e_2^i = e_1^i$ for all $j_1 \leqslant i \leqslant n$ and variable x_{j_1} being simultaneously nonmultiplicative for monomial u_1 and multiplicative for monomial u_2. Thus $e_2^{j_1} = e_1^{j_1} + 1$. It follows that $u_1 < u_2$ in the InvLex monomial ordering (see Section 1.2.1) and so, by induction, $u_1 < u_2 < \cdots < u_m$ in the InvLex monomial ordering. The monomials in the sequence $\{u_1, u_2, \ldots, u_m\}$ are therefore all different. □

Proposition 4.3.3 *If an involutive division I is continuous, and a given set of polynomials P is a Locally Involutive Basis with respect to I and some admissible monomial ordering O, then P is an Involutive Basis with respect to I and O.*

Proof: Let I be a continuous involutive division; let O be an admissible monomial ordering; and let P be a Locally Involutive Basis with respect to I and O. Given any polynomial $p \in P$ and any term t, in order to show that P is an Involutive Basis with respect to I and O, we must show that $pt \xrightarrow[I]{}_P 0$.

If $p \mid_I pt$ we are done, as we can use p to involutively reduce pt to obtain a zero remainder. Otherwise, $\exists\, y_1 \notin \mathcal{M}_I(\mathrm{LM}(p), \mathrm{LM}(P))$ such that t contains y_1. By Local Involutivity, the prolongation py_1 involutively reduces to zero using P. Assuming that the first step of this involutive reduction involves the polynomial $p_1 \in P$, we can write

$$py_1 = p_1 t_1 + \sum_{a=1}^{A} p_{\alpha_a} t_{\alpha_a}, \tag{4.3}$$

where $p_{\alpha_a} \in P$ and t_1, t_{α_a} are terms which are multiplicative (over P) for p_1 and each p_{α_a} respectively. Multiplying both sides of Equation (4.3) by $\frac{t}{y_1}$, we obtain the equation

$$pt = p_1 t_1 \frac{t}{y_1} + \sum_{a=1}^{A} p_{\alpha_a} t_{\alpha_a} \frac{t}{y_1}. \tag{4.4}$$

If $p_1 \mid_I pt$, it is clear that we can use p_1 to involutively reduce the polynomial pt to obtain the polynomial $\sum_{a=1}^{A} p_{\alpha_a} t_{\alpha_a} \frac{t}{y_1}$. By Proposition 4.2.4, we can then continue to involutively reduce pt by repeating this proof on each polynomial $p_{\alpha_a} t_{\alpha_a} \frac{t}{y_1}$ individually (where $1 \leqslant a \leqslant A$), noting that this process will terminate because of the admissibility of O (we have $\mathrm{LM}(p_{\alpha_a} t_{\alpha_a} \frac{t}{y_1}) < \mathrm{LM}(pt)$ for all $1 \leqslant a \leqslant A$).

Otherwise, if p_1 does not involutively divide pt, there exists a variable $y_2 \in \frac{t}{y_1}$ such that $y_2 \notin \mathcal{M}_I(\mathrm{LM}(p_1), \mathrm{LM}(P))$. By Local Involutivity, the prolongation $p_1 y_2$ involutively reduces to zero using P. Assuming that the first step of this involutive reduction involves the polynomial $p_2 \in P$, we can write

$$p_1 y_2 = p_2 t_2 + \sum_{b=1}^{B} p_{\beta_b} t_{\beta_b}, \tag{4.5}$$

where $p_{\beta_b} \in P$ and t_2, t_{β_b} are terms which are multiplicative (over P) for p_2 and each p_{β_b} respectively. Multiplying both sides of Equation (4.5) by $\frac{t_1 t}{y_1 y_2}$, we obtain the equation

$$p_1 t_1 \frac{t}{y_1} = p_2 t_2 \frac{t_1 t}{y_1 y_2} + \sum_{b=1}^{B} p_{\beta_b} t_{\beta_b} \frac{t_1 t}{y_1 y_2}. \tag{4.6}$$

Substituting for $p_1 t_1 \frac{t}{y_1}$ from Equation (4.6) into Equation (4.4), we obtain the equation

$$pt = p_2 t_2 \frac{t_1 t}{y_1 y_2} + \sum_{a=1}^{A} p_{\alpha_a} t_{\alpha_a} \frac{t}{y_1} + \sum_{b=1}^{B} p_{\beta_b} t_{\beta_b} \frac{t_1 t}{y_1 y_2}. \tag{4.7}$$

If $p_2 \mid_I pt$, it is clear that we can use p_2 to involutively reduce the polynomial pt to obtain the polynomial $\sum_{a=1}^{A} p_{\alpha_a} t_{\alpha_a} \frac{t}{y_1} + \sum_{b=1}^{B} p_{\beta_b} t_{\beta_b} \frac{t_1 t}{y_1 y_2}$. As before, we can then use Proposition 4.2.4 to continue the involutive reduction of pt by repeating this proof on each summand individually.

Otherwise, if p_2 does not involutively divide pt, we continue by induction, obtaining a sequence $p, p_1, p_2, p_3, \ldots$ of elements in P. By construction, each element in the sequence divides pt. By continuity, each element in the sequence is different. Because P is finite and because pt has a finite number of distinct divisors, the sequence must be finite, terminating with an involutive divisor $p' \in P$ of pt, which then allows us to finish the proof through use of Proposition 4.2.4 and the admissibility of O. □

Remark 4.3.4 The above proof is a slightly clarified version of the proof of Theorem 6.5 in [25].

Definition 4.3.5 (Constructivity) Let I be an involutive division, and let U be an arbitrary set of monomials over a polynomial ring $R[x_1, \ldots, x_n]$. We say that I is *constructive* if, given any monomial $u \in U$ and any nonmultiplicative variable $x_i \notin \mathcal{M}_I(u, U)$ satisfying the following two conditions, no monomial $w \in \mathcal{C}_I(U)$ exists such that $ux_i \in \mathcal{C}_I(w, U \cup \{w\})$.

(a) $ux_i \notin \mathcal{C}_I(U)$.

(b) If there exists a monomial $v \in U$ and a nonmultiplicative variable $x_j \notin \mathcal{M}_I(v, U)$ such that $vx_j \mid ux_i$ but $vx_j \neq ux_i$, then $vx_j \in \mathcal{C}_I(U)$.

Remark 4.3.6 Constructivity allows us to consider only polynomials whose lead monomials lie *outside* the current involutive span as potential new Involutive Basis elements.

Proposition 4.3.7 *The Thomas, Pommaret and Janet involutive divisions are all constructive.*

Proof: Throughout, let the monomials u, v and w that appear in Definition 4.3.5 have corresponding multidegrees $(e_u^1, e_u^2, \ldots, e_u^n)$, $(e_v^1, e_v^2, \ldots, e_v^n)$ and $(e_w^1, e_w^2, \ldots, e_w^n)$; and let the monomials w_1, w_2, w_3 and μ that appear in this proof have corresponding multidegrees $(e_{w_1}^1, e_{w_1}^2, \ldots, e_{w_1}^n)$, $(e_{w_2}^1, e_{w_2}^2, \ldots, e_{w_2}^n)$, $(e_{w_3}^1, e_{w_3}^2, \ldots, e_{w_3}^n)$ and $(e_\mu^1, e_\mu^2, \ldots, e_\mu^n)$.

To prove that a particular involutive division I is constructive, we will assume that a monomial $w \in \mathcal{C}_I(U)$ exists such that $ux_i \in \mathcal{C}_I(w, U \cup \{w\})$. Then $w = \mu w_1$ for some

monomial $\mu \in U$ and some monomial w_1 that is multiplicative for μ over the set U ($e^k_{w_1} > 0 \Rightarrow x_k \in \mathcal{M}_I(\mu, U)$ for all $1 \leqslant k \leqslant n$); and $ux_i = ww_2$ for some monomial w_2 that is multiplicative for w over the set $U \cup \{w\}$ ($e^k_{w_2} > 0 \Rightarrow x_k \in \mathcal{M}_I(w, U \cup \{w\})$ for all $1 \leqslant k \leqslant n$). It follows that $ux_i = \mu w_1 w_2$. If we can show that all variables appearing in w_2 are multiplicative for μ over the set U ($e^k_{w_2} > 0 \Rightarrow x_k \in \mathcal{M}_I(\mu, U)$ for all $1 \leqslant k \leqslant n$), then μ is an involutive divisor of ux_i, contradicting the assumption $ux_i \notin \mathcal{C}_I(U)$.

Thomas. Let x_k be an arbitrary variable ($1 \leqslant k \leqslant n$) such that $e^k_{w_2} > 0$. If $e^k_{w_1} > 0$, then it is clear that x_k is multiplicative for μ. Otherwise $e^k_{w_1} = 0$ so that $e^k_w = e^k_\mu$. By definition, this implies that $x_k \in \mathcal{M}_\mathcal{T}(\mu, U)$ as $x_k \in \mathcal{M}_\mathcal{T}(w, U \cup \{w\})$. Thus $x_k \in \mathcal{M}_\mathcal{T}(\mu, U)$.

Pommaret. Let α and β ($1 \leqslant \alpha, \beta \leqslant n$) be the smallest integers such that $e^\alpha_\mu > 0$ and $e^\beta_w > 0$ respectively. By definition, $\beta \leqslant \alpha$ (because $w = \mu w_1$), so for an arbitrary $1 \leqslant k \leqslant n$, it follows that $e^k_{w_2} > 0 \Rightarrow k \leqslant \beta \leqslant \alpha \Rightarrow x_k \in \mathcal{M}_\mathcal{P}(\mu, U)$ as required.

Janet. Here we proceed by searching for a monomial $\nu \in U$ such that $ux_i \in \mathcal{C}_\mathcal{J}(\nu, U)$, contradicting the assumption $ux_i \notin \mathcal{C}_\mathcal{J}(U)$. Let α and β ($1 \leqslant \alpha, \beta \leqslant n$) be the largest integers such that $e^\alpha_{w_1} > 0$ and $e^\beta_{w_2} > 0$ respectively (such integers will exist because if $\deg(w_1) = 0$ or $\deg(w_2) = 0$, we obtain an immediate contradiction $ux_i \in \mathcal{C}_\mathcal{J}(U)$). We claim that $i > \max\{\alpha, \beta\}$.

- If $i < \beta$, then $e^\beta_w < e^\beta_u$ which contradicts $x_\beta \in \mathcal{M}_\mathcal{J}(w, U \cup \{w\})$ as $e^\gamma_w = e^\gamma_u$ for all $\gamma > \beta$. Thus $i \geqslant \beta$.
- If $i < \alpha$, then as $\beta \leqslant i$ we must have $e^\gamma_\mu = e^\gamma_u$ for all $\alpha < \gamma \leqslant n$. Therefore $e^\alpha_\mu < e^\alpha_u \Rightarrow x_\alpha \notin \mathcal{M}_\mathcal{J}(\mu, U)$, a contradiction; it follows that $i \geqslant \alpha$.
- If $i = \alpha$, then either $\beta < \alpha$ or $\beta = \alpha$. If $\beta = \alpha$, then as $e^i_{w_1} > 0$; $e^i_{w_2} > 0$ and $e^i_u + 1 = e^i_\mu + e^i_{w_1} + e^i_{w_2}$, we have $e^i_u > e^i_\mu \Rightarrow x_\alpha \notin \mathcal{M}_\mathcal{J}(\mu, U)$, a contradiction. If $\beta < \alpha$, then $e^i_u + 1 = e^i_\mu + e^i_{w_1}$. If $e^i_{w_1} \geqslant 2$, we get the same contradiction as before ($x_\alpha \notin \mathcal{M}_\mathcal{J}(\mu, U)$). Otherwise $e^i_{w_1} = 1$ so that $e^\gamma_u = e^\gamma_\mu$ for all $\alpha \leqslant \gamma \leqslant n$. If $w = \mu x_i$, then as $e^\beta_w < e^\beta_u$ we have $x_\beta \notin \mathcal{M}_\mathcal{J}(w, U \cup \{w\})$, a contradiction. Else let δ (where $1 \leqslant \delta < \alpha$) be the second greatest integer such that $e^\delta_{w_1} > 0$. Then, as $e^\delta_\mu < e^\delta_u$ and $e^\gamma_\mu = e^\gamma_u$ for all $\delta < \gamma \leqslant n$, we have $x_\delta \notin \mathcal{M}_\mathcal{J}(\mu, U)$, another contradiction. It follows that $i > \max\{\alpha, \beta\}$, so that $e^\gamma_u = e^\gamma_\mu$ for all $i < \gamma \leqslant n$ and $e^i_u + 1 = e^i_\mu$.

If $ux_i \notin \mathcal{C}_\mathcal{J}(U)$, then there must exist a variable x_k (where $1 \leqslant k < i$) such that $e^k_{w_2} > 0$ and $x_k \notin \mathcal{M}_\mathcal{J}(\mu, U)$. Because $e^\alpha_{w_1} > 0$, we can use condition (b) of Definition 4.3.5 to give

us a monomial $\mu_1 \in U$ and a monomial w_3 multiplicative for μ_1 over U ($e^\gamma_{w_3} > 0 \Rightarrow x_\gamma \in \mathcal{M}_\mathcal{J}(\mu_1, U)$ for all $1 \leqslant \gamma \leqslant n$) such that

$$\begin{aligned} ux_i &= \mu w_1 w_2 \\ &= \mu x_k w_1 \left(\frac{w_2}{x_k}\right) \\ &= \mu_1 w_3 w_1 \left(\frac{w_2}{x_k}\right). \end{aligned}$$

If $\mu_1 \mid_\mathcal{J} ux_i$, then the proof is complete, with $\nu = \mu_1$. Otherwise there must be a variable $x_{k'}$ appearing in the monomial $w_1(\frac{w_2}{x_k})$ such that $x_{k'} \notin \mathcal{M}_\mathcal{J}(\mu_1, U)$. To use condition (b) of Definition 4.3.5 to yield a monomial $\mu_2 \in U$ and a monomial w_4 multiplicative for μ_2 over U such that

$$\mu_1 w_3 w_1 \left(\frac{w_2}{x_k}\right) = \mu_2 w_4 \left(\frac{w_1 w_2}{x_k x_{k'}}\right) w_3,$$

it is sufficient to demonstrate that at least one variable appearing in the monomial $w_3 w_1(\frac{w_2}{x_k})$ is multiplicative for μ_1 over the set U. We will do this by showing that $x_\alpha \in \mathcal{M}_\mathcal{J}(\mu_1, U)$ (recall that $e^\alpha_{w_1} > 0$).

By the definition of the Janet involutive division,

$$e^\gamma_{\mu_1} = e^\gamma_\mu \text{ for all } k < \gamma \leqslant n \tag{4.8}$$

and

$$e^k_{\mu_1} = e^k_\mu + 1, \tag{4.9}$$

so that $\mu < \mu_1$ in the InvLex monomial ordering. If we can show that $\alpha > k$, then it is clear from Equation (4.8) and $x_\alpha \in \mathcal{M}_\mathcal{J}(\mu, U)$ that $x_\alpha \in \mathcal{M}_\mathcal{J}(\mu_1, U)$.

- If $\alpha > \beta$, then $\alpha > k$ because $k \leqslant \beta$ by definition.
- If $\alpha = \beta$, then $\alpha > k$ if $k < \beta$; otherwise $k = \beta$ in which case $x_\alpha \in \mathcal{M}_\mathcal{J}(\mu, U)$ is contradicted by Equations (4.8) and (4.9).
- If $\alpha < \beta$, then $e^\gamma_\mu = e^\gamma_w$ for all $\alpha < \gamma \leqslant n$. Thus $k \leqslant \alpha$ otherwise $x_k \in \mathcal{M}_\mathcal{J}(w, U \cup \{w\}) \Rightarrow x_k \in \mathcal{M}_\mathcal{J}(\mu, U)$, a contradiction. Further, $k = \alpha$ is not allowed because $x_\alpha \in \mathcal{M}_\mathcal{J}(\mu, U)$ and $x_k \notin \mathcal{M}_\mathcal{J}(\mu, U)$ cannot both be true; therefore $\alpha > k$ again.

If $\mu_2 \mid_{\mathcal{J}} ux_i$, then the proof is complete, with $\nu = \mu_2$. Otherwise we proceed by induction to obtain the sequence shown below (Equation (4.10)), which is valid because $\mu_{\sigma-1} < \mu_\sigma$ (for $\sigma \geqslant 2$) in the InvLex monomial ordering allows us to prove that the variable x_α (that appears in the monomial w_1) is multiplicative (over U) for the monomial μ_σ; this in turn enables us to construct the next entry in the sequence by using condition (b) of Definition 4.3.5.

$$\mu w_1 w_2 = \mu_1 w_3 w_1 \left(\frac{w_2}{x_k}\right) = \mu_2 w_4 \left(\frac{w_1 w_2}{x_k x_{k'}}\right) w_3 = \mu_3 w_5 \left(\frac{w_1 w_2 w_3}{x_k x_{k'} x_{k''}}\right) w_4 = \cdots \tag{4.10}$$

Because $\mu < \mu_1 < \mu_2 < \cdots$ in the InvLex monomial ordering, elements of the sequence $\mu, \mu_1, \mu_2, \ldots$ are distinct. It follows that the sequence in Equation (4.10) is finite (terminating with the required ν) because μ and the μ_σ (for $\sigma \geqslant 1$) are all divisors of the monomial ux_i, of which there are only a finite number of. $\square$

Remark 4.3.8 The above proof that Janet is a constructive involutive division does not use the property of Janet being a continuous involutive division, unlike the proofs found in [25] and [50].

4.4 The Involutive Basis Algorithm

To compute an Involutive Basis for an ideal J with respect to some admissible monomial ordering O and some involutive division I, it is sufficient to compute a Locally Involutive Basis for J with respect to I and O if I is continuous; and we can compute this Locally Involutive Basis by considering only prolongations whose lead monomials lie outside the current involutive span if I is constructive. Let us now consider Algorithm 9, an algorithm to construct an Involutive Basis for J (with respect to I and O) in exactly this manner.

The algorithm starts by autoreducing the input basis F using Algorithm 8. We then construct a set S containing all the possible prolongations of elements of F, before recursively (a) picking a polynomial s from S such that $\mathrm{LM}(s)$ is minimal in the chosen monomial ordering; (b) removing s from S; and (c) finding the involutive remainder s' of s with respect to F.

If during this loop a remainder s' is found that is nonzero, we exit the loop and autoreduce the set $F \cup \{s'\}$, continuing thereafter to construct a new set S and repeating the above process on this new set. If however all the prolongations in S involutively reduce to zero,

Algorithm 9 The Commutative Involutive Basis Algorithm

Input: A Basis $F = \{f_1, f_2, \ldots, f_m\}$ for an ideal J over a commutative polynomial ring $R[x_1, \ldots x_n]$; an admissible monomial ordering O; a continuous and constructive involutive division I.

Output: An Involutive Basis $G = \{g_1, g_2, \ldots, g_p\}$ for J (in the case of termination).

$G = \emptyset$;
$F = \text{Autoreduce}(F)$;
while $(G == \emptyset)$ **do**
  $S = \{x_i f \mid f \in F,\ x_i \notin \mathcal{M}_I(f, F)\}$;
  $s' = 0$;
  while $(S \neq \emptyset)$ **and** $(s' == 0)$ **do**
    Let s be a polynomial in S whose lead monomial is minimal with respect to O;
    $S = S \setminus \{s\}$;
    $s' = \text{Rem}_I(s, F)$;
  end while
  if $(s' \neq 0)$ **then**
    $F = \text{Autoreduce}(F \cup \{s'\})$;
  else
    $G = F$;
  end if
end while
return G;

then by definition F is a Locally Involutive Basis, and so we can exit the algorithm with this basis. The correctness of Algorithm 9 is therefore clear; termination however requires us to show that each involutive division used with the algorithm is *Noetherian* and *stable*.

Definition 4.4.1 An involutive division I is *Noetherian* if, given any finite set of monomials U, there is a finite Involutive Basis $V \supseteq U$ with respect to I and some arbitrary admissible monomial ordering O.

Proposition 4.4.2 *The Thomas and Janet divisions are Noetherian.*

Proof: Let $U = \{u_1, \ldots, u_m\}$ be an arbitrary set of monomials over a polynomial ring $\mathcal{R} = R[x_1, \ldots, x_n]$ generating an ideal J. We will explicitly construct an Involutive Basis V for U with respect to some arbitrary admissible monomial ordering O.

Janet (Adapted from [50], Lemma 2.13). Let $\mu \in \mathcal{R}$ be the monomial with multidegree $(e^1_\mu, e^2_\mu, \ldots, e^n_\mu)$ defined as follows: $e^i_\mu = \max_{u \in U} e^i_u$ $(1 \leqslant i \leqslant n)$. We claim that the set V containing all monomials $v \in J$ such that $v \mid \mu$ is an Involutive Basis for U with respect to the Janet involutive division and O. To prove the claim, first note that V is a basis for J because $U \subseteq V$ and $V \subset J$; to prove that V is a Janet Involutive Basis for J we have to show that all multiples of elements of V involutively reduce to zero using V, which we shall do by showing that all members of the ideal involutively reduce to zero using V.

Let p be an arbitrary element of J. If $p \in V$, then trivially $p \in \mathcal{C}_\mathcal{J}(V)$ and so p involutively reduces to zero using V. Otherwise set $X = \{x_i \text{ such that } e^i_{\mathrm{LM}(p)} > e^i_\mu\}$, and define the monomial p' by $e^i_{p'} = e^i_{\mathrm{LM}(p)}$ for $x_i \notin X$; and $e^i_{p'} = e^i_\mu$ for $x_i \in X$ (so that $e^i_{p'} = \min\{e^i_{\mathrm{LM}(p)}, e^i_\mu\}$). By construction of the set V and by the definition of μ, it follows that $v' \in V$ and $X \subseteq \mathcal{M}_\mathcal{J}(p', V)$. But this implies that $\mathrm{LM}(p) \in \mathcal{C}_\mathcal{J}(p', V)$, and thus $p \xrightarrow[\mathcal{J}]{}_{p'} (p - \mathrm{LM}(p))$. By induction and by the admissibility of O, $p \xrightarrow[\mathcal{J}]{}_V 0$ and thus V is a finite Janet Involutive Basis for J.

Thomas. We use the same proof as for Janet above, replacing "Janet" by "Thomas" and "$\mathcal{J}$" by "$\mathcal{T}$". □

Proposition 4.4.3 *The Pommaret division is not Noetherian.*

Proof: Let J be the ideal generated by the monomial $u := xy$ over the polynomial ring $\mathbb{Q}[x, y]$. For the Pommaret division, $\mathcal{M}_\mathcal{P}(u) = \{x\}$, and it is clear that $\mathcal{M}_\mathcal{P}(v) = \{x\}$ for

all $v \in J$ as $v \in J \Rightarrow v = (xy)p$ for some polynomial p. It follows that no finite Pommaret Involutive Basis exists for J as no prolongation by the variable y of any polynomial $p \in J$ is involutively divisible by some other polynomial $p' \in J$; the Pommaret Involutive Basis for J is therefore the infinite basis $\{xy,\ xy^2,\ xy^3,\ \ldots\}$. □

Definition 4.4.4 Let u and v be two distinct monomials such that $u \mid v$. An involutive division I is *stable* if $\mathrm{Rem}_I(v, \{u, v\}, \{u\}) = v$. In other words, u is not an involutive divisor of v with respect to I when multiplicative variables are taken over the set $\{u, v\}$.

Proposition 4.4.5 *The Thomas and Janet divisions are stable.*

Proof: Let u and v have corresponding multidegrees $(e_u^1, \ldots, e_u^n)$ and $(e_v^1, \ldots, e_v^n)$. If $u \mid v$ and if u and v are different, then we must have $e_u^i < e_v^i$ for at least one $1 \leqslant i \leqslant n$.

Thomas. By definition, $x_i \notin \mathcal{M}_{\mathcal{T}}(u, \{u, v\})$, so that $\mathrm{Rem}_{\mathcal{T}}(v, \{u, v\}, \{u\}) = v$.

Janet. Let j be the greatest integer such that $e_u^j < e_v^j$. Then, as $e_u^k = e_v^k$ for all $j < k \leqslant n$, it follows that $x_j \notin \mathcal{M}_{\mathcal{J}}(u, \{u, v\})$, and so $\mathrm{Rem}_{\mathcal{J}}(v, \{u, v\}, \{u\}) = v$. □

Proposition 4.4.6 *The Pommaret division is not stable.*

Proof: Consider the two monomials $u := x$ and $v := x^2$ over the polynomial ring $\mathbb{Q}[x]$. Because $\mathcal{M}_{\mathcal{P}}(u, \{u, v\}) = \{x\}$, it is clear that $u \mid_{\mathcal{P}} v$, and so the Pommaret involutive division is not stable. □

Remark 4.4.7 Stability ensures that any set of distinct monomials is autoreduced. In particular, if a set U of monomials is autoreduced, and if we add a monomial $u \notin U$ to U, then the resultant set $U \cup \{u\}$ is also autoreduced. This contradicts a statement made on page 24 of [50], where it is claimed that if we add an involutively irreducible prolongation ux_i of a monomial u from an autoreduced set of monomials U to that set, then the resultant set is also autoreduced regardless of whether or not the involutive division used is stable[2]. For a counterexample, consider the set of monomials $U := \{u_1, u_2\} = \{xy, x^2y^2\}$ over the polynomial ring $\mathbb{Q}[x, y]$, and let the involutive division be Pommaret.

u	$\mathcal{M}_{\mathcal{P}}(u, U)$
xy	$\{x\}$
x^2y^2	$\{x\}$

[2]This claim is integral to the proof of Theorem 6.4 in [50], a theorem that states than an algorithm corresponding to Algorithm 9 in this thesis terminates.

Because the variable y is nonmultiplicative for the monomial xy, it is clear that the set U is autoreduced. Consider the prolongation xy^2 of the monomial u_1 by the variable y. This prolongation is involutively irreducible with respect to U, but if we add the prolongation to U to obtain the set $V := \{v_1, v_2, v_3\} = \{xy, x^2y^2, xy^2\}$, then v_3 will involutively reduce v_2, contradicting the claim that the set V is autoreduced.

v	$\mathcal{M}_{\mathcal{P}}(v, V)$
xy	$\{x\}$
x^2y^2	$\{x\}$
xy^2	$\{x\}$

Proposition 4.4.8 *Algorithm 9 always terminates when used with a Noetherian and stable involutive division.*

Proof: Let I be a Noetherian and stable involutive division, and consider the computation (using Algorithm 9) of an Involutive Basis for a set of polynomials F with respect to I and some admissible monomial ordering O. The algorithm begins by autoreducing F to give a basis (which we shall denote by F_1) generating the same ideal J as F. Each pass of the algorithm then produces a basis $F_{i+1} = \text{Autoreduce}(F_i \cup \{s_i'\})$ generating J ($i \geqslant 1$), where each $s_i' \neq 0$ is an involutively reduced prolongation. Consider the monomial ideal $\langle \text{LM}(F_i) \rangle$ generated by the lead monomials of the set F_i. **Claim:**

$$\langle \text{LM}(F_1) \rangle \subseteq \langle \text{LM}(F_2) \rangle \subseteq \langle \text{LM}(F_3) \rangle \subseteq \cdots \tag{4.11}$$

is an ascending chain of monomial ideals.

Proof of Claim: It is sufficient to show that if an arbitrary polynomial $f \in F_i$ does not appear in F_{i+1}, then there must be a polynomial $f' \in F_{i+1}$ such that $\text{LM}(f') \mid \text{LM}(f)$. It is clear that such an f' will exist if the lead monomial of f is not reduced during autoreduction; otherwise a polynomial p reduces the lead monomial of f during autoreduction, so that $\text{LM}(p) \mid_I \text{LM}(f)$. If there exists a polynomial $p' \in F_{i+1}$ such that $\text{LM}(p') = \text{LM}(p)$, we are done; otherwise we proceed by induction on p to obtain a polynomial q such that $\text{LM}(q) \mid_I \text{LM}(p)$. Because $\deg(\text{LM}(f)) > \deg(\text{LM}(p)) > \deg(\text{LM}(q)) > \cdots$, this process is guaranteed to terminate with the required f'. □

By the Ascending Chain Condition (Corollary 2.2.6), the chain in Equation (4.11) must

eventually become constant, so there must be an integer N ($N \geqslant 1$) such that

$$\langle \mathrm{LM}(F_N)\rangle = \langle \mathrm{LM}(F_{N+1})\rangle = \cdots .$$

Claim: If $F_{k+1} = \mathrm{Autoreduce}(F_k \cup \{s'_k\})$ for some $k \geqslant N$, then $\mathrm{LM}(s'_k) = \mathrm{LM}(fx_i)$ for some polynomial $f \in F_k$ and some variable $x_i \notin \mathcal{M}_I(f, F_k)$ such that $s'_k = \mathrm{Rem}_I(fx_i, F_k)$.

Proof of Claim: Assume to the contrary that $\mathrm{LM}(s'_k) \neq \mathrm{LM}(fx_i)$. Then because $s'_k = \mathrm{Rem}_I(fx_i, F_k)$, it follows that $\mathrm{LM}(s'_k) < \mathrm{LM}(fx_i)$. But $\langle \mathrm{LM}(F_k)\rangle = \langle \mathrm{LM}(F_{k+1})\rangle$, so that $\mathrm{LM}(s'_k) = \mathrm{LM}(f'u)$ for some $f' \in F_k$ and some monomial u containing at least one variable $x_j \notin \mathcal{M}_I(f', F_k)$ (otherwise s'_k can be involutively reduced with respect to F_k, a contradiction).

Because O is admissible, $1 \leqslant \frac{u}{x_j}$ and therefore $x_j \leqslant u$, so that $\mathrm{LM}(f'x_j) \leqslant \mathrm{LM}(f'u) < \mathrm{LM}(fx_i)$. But the prolongation fx_i was chosen so that its lead monomial is minimal amongst the lead monomials of all prolongations of elements of F_k that do not involutively reduce to zero; the prolongation $f'x_k$ must therefore involutively reduce to zero, so that $\mathrm{LM}(f'x_j) = \mathrm{LM}(f''u')$ for some polynomial $f'' \in F_k$ and some monomial u' that is multiplicative for f'' over F_k. But s'_k is involutively irreducible with respect to F_k, so a variable $x'_j \notin \mathcal{M}_I(f'', F_k)$ must appear in the monomial $\frac{u}{x_j}$.

It is now clear that we can construct a sequence $f'x_j$, $f''x'_j, \ldots$ of prolongations. But I is continuous, so all elements in the corresponding sequence $\mathrm{LM}(f')$, $\mathrm{LM}(f''), \ldots$ of monomials must be distinct. Because F_k is finite, it follows that the sequence of prolongations will terminate with a prolongation that does not involutively reduce to zero and whose lead monomial is less than the monomial $\mathrm{LM}(fx_i)$, contradicting our assumptions. Thus $\mathrm{LM}(s'_k)$ for $k \geqslant N$ is always equal to the lead monomial of some prolongation of some polynomial $f \in F_k$. □

Consider now the set of monomials $\mathrm{LM}(F_{k+1})$. **Claim:** $\mathrm{LM}(F_{k+1}) = \mathrm{LM}(F_k) \cup \{\mathrm{LM}(s'_k)\}$ for all $k \geqslant N$, so that when autoreducing the set $F_k \cup \{s'_k\}$, no leading monomial is involutively reducible.

Proof of Claim: Consider an arbitrary polynomial $p \in F_k \cup \{s'_k\}$. If $p = s'_k$, then (by definition) p is irreducible with respect to the set F_k, and so (by condition (b) of Definition 4.1.4) p will also be irreducible with respect to the set $F_k \cup \{s'_k\}$. If $p \neq s'_k$, then p is irreducible with respect to the set F_k (as the set F_k is autoreduced), and so (again by condition (b) of Definition 4.1.4) the only polynomial in the set $F_k \cup \{s'_k\}$

that can involutively reduce the polynomial p is the polynomial s'_k. But I is stable, so that s'_k cannot involutively reduce $\mathrm{LM}(p)$. It follows that a polynomial p' will appear in the autoreduced set F_{k+1} such that $\mathrm{LM}(p') = \mathrm{LM}(p)$, and thus $\mathrm{LM}(F_{k+1}) = \mathrm{LM}(F_k) \cup \{\mathrm{LM}(s'_k)\}$ as required. □

For the final part of the proof, consider the basis F_N. Because I is Noetherian, there exists a finite Involutive Basis G_N for the ideal generated by the set of lead monomials $\mathrm{LM}(F_N)$, where $G_N \supseteq \mathrm{LM}(F_N)$. Let fx_i be the prolongation chosen during the N-th iteration of Algorithm 9, so that $\mathrm{LM}(fx_i) \notin \mathcal{C}_I(F_N)$. Because G_N is an Involutive Basis for $\mathrm{LM}(F_N)$, there must be a monomial $g \in G_N$ such that $g \mid_I \mathrm{LM}(fx_i)$. **Claim:** $g = \mathrm{LM}(fx_i)$.

Proof of Claim: We proceed by showing that if $g \neq \mathrm{LM}(fx_i)$, then $g \in \mathcal{C}_I(\mathrm{LM}(F_N))$ so that (because of condition (b) of Definition 4.1.4) $\mathrm{LM}(fx_i) \in \mathcal{C}_I(G_N) \Rightarrow \mathrm{LM}(fx_i) \in \mathcal{C}_I(g, \mathrm{LM}(F_N) \cup \{g\})$, contradicting the constructivity of I (Definition 4.3.5).

Assume that $g \neq \mathrm{LM}(fx_i)$. Because $\langle G_N \rangle = \langle \mathrm{LM}(F_N) \rangle$, there exists a polynomial $f_1 \in F_N$ such that $\mathrm{LM}(f_1) \mid g$. If $\mathrm{LM}(f_1) \mid_I g$ with respect to F_N, then we are done. Otherwise $\mathrm{LM}(g) = \mathrm{LM}(f_1)u_1$ for some monomial $u_1 \neq 1$ containing at least one variable $x_{j_1} \notin \mathcal{M}_I(f_1, F_N)$. Because $\deg(g) < \deg(\mathrm{LM}(fx_i))$ and $\mathrm{LM}(f_1)x_{j_1} \mid \mathrm{LM}(fx_i)$, we must have $\mathrm{LM}(f_1)x_{j_1} < \mathrm{LM}(fx_i)$ with respect to our chosen monomial ordering, so that $\mathrm{LM}(f_1)x_{j_1} \in \mathcal{C}_I(F_N)$ by definition of how the prolongation fx_i was chosen. It follows that there exists a polynomial $f_2 \in F_N$ such that $\mathrm{LM}(f_2) \mid_I \mathrm{LM}(f_1)x_{j_1}$ with respect to F_N. If $\mathrm{LM}(f_2) \mid_I g$ with respect to F_N, then we are done. Otherwise we iterate ($\mathrm{LM}(f_1)x_{j_1} = \mathrm{LM}(f_2)u_2$ for some monomial u_2 containing at least one variable $x_{j_2} \notin \mathcal{M}_I(f_2, F_N) \ldots$) to obtain the sequence $(f_1, f_2, f_3, \ldots)$ of polynomials, where the lead monomial of each element in the sequence divides g and $\mathrm{LM}(f_{k+1}) \mid_I \mathrm{LM}(f_k)x_{j_k}$ with respect to F_N for all $k \geqslant 1$. Because I is continuous, this sequence must be finite, terminating with a polynomial $f_k \in F_N$ (for some $k \geqslant 1$) such that $f_k \mid_I g$ with respect to F_N. □

It follows that during the N-th iteration of the algorithm, a polynomial is added to the current basis F_N whose lead monomial is a member of the Involutive Basis G_N. By induction, every step of the algorithm after the N-th step also adds a polynomial to the current basis whose lead monomial is a member of G_N. Because G_N is a finite set, after a finite number of steps the basis $\mathrm{LM}(F_k)$ (for some $k \geqslant N$) will contain all the elements of G_N. We can therefore deduce that $\mathrm{LM}(F_k) = G_N$; it follows that $\mathrm{LM}(F_k)$ is an Involutive Basis, and so F_k is also an Involutive Basis. □

Theorem 4.4.9 *Every Involutive Basis is a Gröbner Basis.*

Proof: Let $G = \{g_1, \ldots, g_m\}$ be an Involutive Basis with respect to some involutive division I and some admissible monomial ordering O, where each $g_i \in G$ (for all $1 \leqslant i \leqslant m$) is a member of the polynomial ring $R[x_1, \ldots, x_n]$. To prove that G is a Gröbner Basis, we must show that all S-polynomials

$$\text{S-pol}(g_i, g_j) = \frac{\text{lcm}(\text{LM}(g_i), \text{LM}(g_j))}{\text{LT}(g_i)} g_i - \frac{\text{lcm}(\text{LM}(g_i), \text{LM}(g_j))}{\text{LT}(g_j)} g_j$$

conventionally reduce to zero using G ($1 \leqslant i, j \leqslant m$, $i \neq j$). Because G is an Involutive Basis, it is clear that $\frac{\text{lcm}(\text{LM}(g_i),\text{LM}(g_j))}{\text{LT}(g_i)} g_i \xrightarrow[I]{}_G 0$ and $\frac{\text{lcm}(\text{LM}(g_i),\text{LM}(g_j))}{\text{LT}(g_j)} g_j \xrightarrow[I]{}_G 0$. By Proposition 4.2.4, it follows that $\text{S-pol}(g_i, g_j) \xrightarrow[I]{}_G 0$. But every involutive reduction is a conventional reduction, so we can deduce that $\text{S-pol}(g_i, g_j) \to_G 0$ as required. $\Box$

Lemma 4.4.10 *Remainders are involutively unique with respect to Involutive Bases.*

Proof: Given an Involutive Basis G with respect to some involutive division I and some admissible monomial ordering O, Theorem 4.4.9 tells us that G is a Gröbner Basis with respect to O and thus remainders are conventionally unique with respect to G. To prove that remainders are involutively unique with respect to G, we must show that the conventional and involutive remainders of an arbitrary polynomial p with respect to G are identical. For this it is sufficient to show that a polynomial p is conventionally reducible by G if and only if it is involutively reducible by G. ($\Rightarrow$) Trivial as every involutive reduction is a conventional reduction. ($\Leftarrow$) If a polynomial p is conventionally reducible by a polynomial $g \in G$, it follows that $\text{LM}(p) = \text{LM}(g)u$ for some monomial u. But G is an Involutive Basis, so there must exist a polynomial $g' \in G$ such that $\text{LM}(g)u = \text{LM}(g')u'$ for some monomial u' that is multiplicative (over G) for g'. Thus p is also involutively reducible by G. $\Box$

Example 4.4.11 Let us return to our favourite example of an ideal J generated by the set of polynomials $F := \{f_1, f_2\} = \{x^2 - 2xy + 3,\ 2xy + y^2 + 5\}$ over the polynomial ring $\mathbb{Q}[x, y, z]$. To compute an Involutive Basis for F with respect to the DegLex monomial ordering and the Janet involutive division $\mathcal{J}$, we apply Algorithm 9 to F, in which the first task is to autoreduce F. This produces the set $F = \{f_2,\ f_3\} = \{2xy + y^2 + 5,\ x^2 + y^2 + 8\}$ as output (because $f_1 = x^2 - 2xy + 3 \xrightarrow[\mathcal{J}]{}_{f_2} x^2 + y^2 + 8 =: f_3$ and f_2 is involutively irreducible with respect to f_3), with multiplicative variables as shown below.

Polynomial	$\mathcal{M}_{\mathcal{J}}(f_i, F)$
$f_2 = 2xy + y^2 + 5$	$\{x, y\}$
$f_3 = x^2 + y^2 + 8$	$\{x\}$

The first set of prolongations of elements of F is the set $S = \{f_3 y\} = \{x^2y + y^3 + 8y\}$. As this set only has one element, it is clear that on entering the second while loop of the algorithm, we must remove the polynomial $s = x^2y + y^3 + 8y$ from S and involutively reduce s with respect to F to give the polynomial $s' = \frac{5}{4}y^3 - \frac{5}{2}x + \frac{37}{4}y$ as follows.

$$\begin{aligned} s = x^2y + y^3 + 8y \quad & \xrightarrow[\mathcal{J}]{} {}_{f_2} \quad x^2y + y^3 + 8y - \frac{1}{2}x(2xy + y^2 + 5) \\ & = \quad -\frac{1}{2}xy^2 + y^3 - \frac{5}{2}x + 8y \\ & \xrightarrow[\mathcal{J}]{} {}_{f_2} \quad -\frac{1}{2}xy^2 + y^3 - \frac{5}{2}x + 8y + \frac{1}{4}y(2xy + y^2 + 5) \\ & = \quad \frac{5}{4}y^3 - \frac{5}{2}x + \frac{37}{4}y = s' =: f_4. \end{aligned}$$

As the prolongation did not involutively reduce to zero, we exit from the second while loop of the algorithm and proceed by autoreducing the set $F \cup \{f_4\} = \{2xy + y^2 + 5,\ x^2 + y^2 + 8,\ \frac{5}{4}y^3 - \frac{5}{2}x + \frac{37}{4}y\}$. This process does not alter the set, so now we consider the prolongations of the three element set $F = \{f_2,\ f_3,\ f_4\}$.

Polynomial	$\mathcal{M}_{\mathcal{J}}(f_i, F)$
$f_2 = 2xy + y^2 + 5$	$\{x\}$
$f_3 = x^2 + y^2 + 8$	$\{x\}$
$f_4 = \frac{5}{4}y^3 - \frac{5}{2}x + \frac{37}{4}y$	$\{x, y\}$

We see that there are 2 prolongations to consider, so that $S = \{f_2 y,\ f_3 y\} = \{2xy^2 + y^3 + 5y,\ x^2y + y^3 + 8y\}$. As $xy^2 < x^2y$ in the DegLex monomial ordering, we must consider the prolongation $f_2 y$ first.

$$\begin{aligned} f_2 y = 2xy^2 + y^3 + 5y \quad & \xrightarrow[\mathcal{J}]{} {}_{f_4} \quad 2xy^2 + y^3 + 5y - \frac{4}{5}\left(\frac{5}{4}y^3 - \frac{5}{2}x + \frac{37}{4}y\right) \\ & = \quad 2xy^2 + 2x - \frac{12}{5}y =: f_5. \end{aligned}$$

As before, the prolongation did not involutively reduce to zero, so now we autoreduce the set $F \cup \{f_5\} = \{2xy + y^2 + 5,\ x^2 + y^2 + 8,\ \frac{5}{4}y^3 - \frac{5}{2}x + \frac{37}{4}y,\ 2xy^2 + 2x - \frac{12}{5}y\}$. Again this leaves the set unchanged, so we proceed with the set $F = \{f_2,\ f_3,\ f_4,\ f_5\}$.

Polynomial	$\mathcal{M}_{\mathcal{J}}(f_i, F)$
$f_2 = 2xy + y^2 + 5$	$\{x\}$
$f_3 = x^2 + y^2 + 8$	$\{x\}$
$f_4 = \frac{5}{4}y^3 - \frac{5}{2}x + \frac{37}{4}y$	$\{x, y\}$
$f_5 = 2xy^2 + 2x - \frac{12}{5}y$	$\{x\}$

This time, $S = \{f_2y,\ f_3y,\ f_5y\} = \{2xy^2 + y^3 + 5y,\ x^2y + y^3 + 8y,\ 2xy^3 + 2xy - \frac{12}{5}y^2\}$, and we must consider the prolongation f_2y first.

$$\begin{aligned}
f_2y = 2xy^2 + y^3 + 5y \quad & \xrightarrow[\mathcal{J}]{} {}_{f_5} \quad 2xy^2 + y^3 + 5y - \left(2xy^2 + 2x - \frac{12}{5}y\right) \\
& = \quad y^3 - 2x + \frac{37}{5}y \\
& \xrightarrow[\mathcal{J}]{} {}_{f_4} \quad y^3 - 2x + \frac{37}{5}y - \frac{4}{5}\left(\frac{5}{4}y^3 - \frac{5}{2}x + \frac{37}{4}y\right) \\
& = \quad 0.
\end{aligned}$$

Because the prolongation involutively reduced to zero, we move on to look at the next prolongation f_3y (which comes from the revised set $S = \{f_3y,\ f_5y\} = \{x^2y + y^3 + 8y,\ 2xy^3 + 2xy - \frac{12}{5}y^2\}$).

$$\begin{aligned}
f_3y = x^2y + y^3 + 8y \quad & \xrightarrow[\mathcal{J}]{} {}_{f_2} \quad x^2y + y^3 + 8y - \frac{1}{2}x(2xy + y^2 + 5) \\
& = \quad -\frac{1}{2}xy^2 + y^3 - \frac{5}{2}x + 8y \\
& \xrightarrow[\mathcal{J}]{} {}_{f_5} \quad -\frac{1}{2}xy^2 + y^3 - \frac{5}{2}x + 8y + \frac{1}{4}\left(2xy^2 + 2x - \frac{12}{5}y\right) \\
& = \quad y^3 - 2x + \frac{37}{5}y \\
& \xrightarrow[\mathcal{J}]{} {}_{f_4} \quad y^3 - 2x + \frac{37}{5}y - \frac{4}{5}\left(\frac{5}{4}y^3 - \frac{5}{2}x + \frac{37}{4}y\right) \\
& = \quad 0.
\end{aligned}$$

Finally, we look at the prolongation f_5y from the set $S = \{2xy^3 + 2xy - \frac{12}{5}y^2\}$.

$$\begin{array}{rcl}
f_5y = 2xy^3 + 2xy - \frac{12}{5}y^2 & \xrightarrow[\mathcal{J}]{}_{f_4} & 2xy^3 + 2xy - \frac{12}{5}y^2 - \frac{8}{5}x\left(\frac{5}{4}y^3 - \frac{5}{2}x + \frac{37}{4}y\right) \\
& = & 4x^2 - \frac{64}{5}xy - \frac{12}{5}y^2 \\
& \xrightarrow[\mathcal{J}]{}_{f_3} & 4x^2 - \frac{64}{5}xy - \frac{12}{5}y^2 - 4(x^2 + y^2 + 8) \\
& = & -\frac{64}{5}xy - \frac{32}{5}y^2 - 32 \\
& \xrightarrow[\mathcal{J}]{}_{f_2} & -\frac{64}{5}xy - \frac{32}{5}y^2 - 32 + \frac{32}{5}(2xy + y^2 + 5) \\
& = & 0.
\end{array}$$

Because this prolongation also involutively reduced to zero using F, we are left with $S = \emptyset$, which means that the algorithm now terminates with the Janet Involutive Basis $G = \{2xy + y^2 + 5,\ x^2 + y^2 + 8,\ \frac{5}{4}y^3 - \frac{5}{2}x + \frac{37}{4}y,\ 2xy^2 + 2x - \frac{12}{5}y\}$ as output.

4.5 Improvements to the Involutive Basis Algorithm

4.5.1 Improved Algorithms

In [58], Zharkov and Blinkov introduced an algorithm for computing an Involutive Basis and proved its termination for zero-dimensional ideals. This work led other researchers to produce improved versions of the algorithm (see for example [4], [13], [23], [26], [27] and [28]); improvements made to the algorithm include the introduction of selection strategies (which, as we have seen in the proof of Proposition 4.4.8, are crucial for proving the termination of the algorithm in general), and the introduction of criteria (analogous to Buchberger's criteria) allowing the *a priori* detection of prolongations that involutively reduce to zero.

4.5.2 Homogeneous Involutive Bases

When computing an Involutive Basis, a prolongation of a homogeneous polynomial is another homogeneous polynomial, and the involutive reduction of a homogeneous polynomial by a set of homogeneous polynomials yields another homogeneous polynomial. It would therefore be entirely feasible for a program computing Involutive Bases for ho-

mogeneous input bases to take advantage of the properties of homogeneous polynomial arithmetic.

It would also be desirable to be able to use such a program on input bases containing non-homogeneous polynomials. The natural way to do this would be to modify the procedure outlined in Definition 2.5.7 by replacing every occurrence of the phrase "a Gröbner Basis" by the phrase "an Involutive Basis", thus creating the following definition.

Definition 4.5.1 Let $F = \{f_1, \ldots, f_m\}$ be a non-homogeneous set of polynomials. To compute an Involutive Basis for F using a program that only accepts sets of homogeneous polynomials as input, we proceed as follows.

(a) Construct a homogeneous set of polynomials $F' = \{h(f_1), \ldots, h(f_m)\}$.

(b) Compute an Involutive Basis G' for F'.

(c) Dehomogenise each polynomial $g' \in G'$ to obtain a set of polynomials G.

Ideally, we would like to say that G is always an Involutive Basis for F as long as the monomial ordering used is extendible, mirroring the conclusion reached in Definition 2.5.7. However, we will only prove the validity of this statement in the case that the set G is autoreduced, and also only for certain combinations of monomial orderings and involutive divisions — all combinations will not work, as the following example demonstrates.

Example 4.5.2 Let $F := \{x_1^2 + x_2^3,\ x_1 + x_3^3\}$ be a basis generating an ideal J over the polynomial ring $\mathbb{Q}[x_1, x_2, x_3]$, and let the monomial ordering be Lex. Computing an Involutive Basis for F with respect to the Janet involutive division using Algorithm 9, we obtain the set $G := \{x_2^3 + x_3^6,\ x_1x_2^2 + x_2^2x_3^3,\ x_1x_2 + x_2x_3^3,\ x_1^2 - x_3^6,\ x_1 + x_3^3\}$.

Taking the homogeneous route, we can homogenise F (with respect to Lex) to obtain the set $F' := \{x_1^2y + x_2^3,\ x_1y^2 + x_3^3\}$ over the polynomial ring $\mathbb{Q}[x_1, x_2, x_3, y]$. Computing an Involutive Basis for F' with respect to the Janet involutive division, we obtain the set $G' := \{x_2^3y^3 + x_3^6,\ x_1x_2^2y^3 + x_2^2x_3^3y,\ x_1x_2y^3 + x_2x_3^3y,\ x_1y^3 + x_3^3y, x_1y^2 + x_3^3, x_1x_3^3y - x_2^3y^2,\ x_1^2x_3^2y + x_2^3x_3^2,\ x_1^2x_3y + x_2^3x_3, x_1^2y + x_2^3,\ x_1x_3^3 - x_2^3y\}$. Finally, if we dehomogenise G', we obtain the set $H := \{x_2^3 + x_3^6,\ x_1x_2^2 + x_2^2x_3^3,\ x_1x_2 + x_2x_3^3,\ x_1 + x_3^3,\ x_1x_3^3 - x_2^3,\ x_1^2x_3^2 + x_2^3x_3^2,\ x_1^2x_3 + x_2^3x_3,\ x_1^2 + x_2^3\}$; however this set is not a Janet Involutive Basis for F, as can be verified by checking that (with respect to H) the variable x_3 is nonmultiplicative for the polynomial $x_2^3 + x_3^6$, and

the prolongation of the polynomial $x_2^3 + x_3^6$ by the variable x_3 is involutively irreducible with respect to H.

The reason why H is not an Involutive Basis for J in the above example is that the Janet multiplicative variables for the set G' do not correspond to the Janet multiplicative variables for the set $H = d(G')$. This means that we cannot use the fact that all prolongations of elements of G' involutively reduce to zero using G' to deduce that all prolongations of elements of H involutively reduce to zero using H. To do this, our involutive division must satisfy the following additional property, which ensures that the multiplicative variables of G' and $d(G')$ do correspond to each other.

Definition 4.5.3 Let O be a fixed extendible monomial ordering. An involutive division I is *extendible with respect to O* if, given any set of polynomials P, we have

$$\mathcal{M}_I(p, P) \setminus \{y\} = \mathcal{M}_I(d(p), d(P))$$

for all $p \in P$, where y is the homogenising variable.

In Section 2.5.2, we saw that of the monomial orderings defined in Section 1.2.1, only Lex, InvLex and DegRevLex are extendible. Let us now consider which involutive divisions are extendible with respect to these three monomial orderings.

Proposition 4.5.4 *The Thomas involutive division is extendible with respect to Lex, InvLex and DegRevLex.*

Proof: Let P be an arbitrary set of polynomials over a polynomial ring containing variables $x_1, \ldots, x_n$ and a homogenising variable y. Because the Thomas involutive division decides whether a variable x_i (for $1 \leqslant i \leqslant n$) is multiplicative for a polynomial $p \in P$ independent of the variable y, it is clear that x_i is multiplicative for p if and only if x_i is multiplicative for $d(p)$ with respect to any of the monomial orderings Lex, InvLex and DegRevLex. It follows that $\mathcal{M}_T(p, P) \setminus \{y\} = \mathcal{M}_T(d(p), d(P))$ as required. □

Proposition 4.5.5 *The Pommaret involutive division is extendible with respect to Lex and DegRevLex.*

Proof: Let p be an arbitrary polynomial over a polynomial ring containing variables $x_1, \ldots, x_n$ and a homogenising variable y. Because we are using either the Lex or the

DegRevLex monomial orderings, the variable y must be lexicographically less than any of the variables $x_1, \ldots, x_n$, and so we can state (without loss of generality) that p belongs to the polynomial ring $R[x_1, \ldots, x_n, y]$. Let $(e^1, e^2, \ldots, e^n, e^{n+1})$ be the multidegree corresponding to the monomial $\mathrm{LM}(p)$, and let $1 \leqslant i \leqslant n+1$ be the smallest integer such that $e^i > 0$.

If $i = n+1$, then the variables $x_1, \ldots, x_n$ will all be multiplicative for p. But then $d(p)$ will be a constant, so that the variables $x_1, \ldots, x_n$ will also all be multiplicative for $d(p)$.

If $i \leqslant n$, then the variables $x_1, \ldots, x_i$ will all be multiplicative for p. But because y is the smallest variable, it is clear that i will also be the smallest integer such that $f^i > 0$, where $(f^1, f^2, \ldots, f^n)$ is the multidegree corresponding to the monomial $\mathrm{LM}(d(p))$. It follows that the variables $x_1, \ldots, x_i$ will also all be multiplicative for $d(p)$, and so we can conclude that $\mathcal{M}_{\mathcal{P}}(p, P) \setminus \{y\} = \mathcal{M}_{\mathcal{P}}(d(p), d(P))$ as required. □

Proposition 4.5.6 *The Pommaret involutive division is not extendible with respect to InvLex.*

Proof: Let $p := yx_2 + x_1^2$ be a polynomial over the polynomial ring $\mathbb{Q}[y, x_1, x_2]$, where y is the homogenising variable (which must be greater than all other variables in order for InvLex to be extendible). As $\mathrm{LM}(p) = yx_2$ with respect to InvLex, it follows that $\mathcal{M}_{\mathcal{P}}(p) = \{y\}$. Further, as $\mathrm{LM}(d(p)) = \mathrm{LM}(x_2 + x_1^2) = x_2$ with respect to InvLex, it follows that $\mathcal{M}_{\mathcal{P}}(d(p)) = \{x_1, x_2\}$. We can now deduce that the Pommaret involutive division is not extendible with respect to InvLex, as $\mathcal{M}_{\mathcal{P}}(p) \setminus \{y\} \neq \mathcal{M}_{\mathcal{P}}(d(p))$, or $\emptyset \neq \{x_1, x_2\}$. □

Proposition 4.5.7 *The Janet involutive division is extendible with respect to InvLex.*

Proof: Let P be an arbitrary set of polynomials over a polynomial ring containing variables $x_1, \ldots, x_n$ and a homogenising variable y. Because we are using the InvLex monomial ordering, the variable y must be lexicographically greater than any of the variables $x_1, \ldots, x_n$, and so we can state (without loss of generality) that p belongs to the polynomial ring $R[y, x_1, \ldots, x_n]$. But the Janet involutive division will then decide whether a variable x_i (for $1 \leqslant i \leqslant n$) is multiplicative for a polynomial $p \in P$ independent of the variable y, so it is clear that x_i is multiplicative for p if and only if x_i is multiplicative for $d(p)$, and so (with respect to InvLex) $\mathcal{M}_{\mathcal{J}}(p, P) \setminus \{y\} = \mathcal{M}_{\mathcal{J}}(d(p), d(P))$ as required. □

Proposition 4.5.8 *The Janet involutive division is not extendible with respect to Lex or DegRevLex.*

Proof: Let $U := \{x_1^2y,\ x_1y^2\}$ be a set of monomials over the polynomial ring $\mathbb{Q}[x_1, y]$, where y is the homogenising variable (which must be less than x_1 in order for Lex and DegRevLex to be extendible). The Janet multiplicative variables for U (with respect to Lex and DegRevLex) are shown in the table below.

u	$\mathcal{M}_{\mathcal{J}}(u, U)$
x_1^2y	$\{x_1\}$
x_1y^2	$\{x_1, y\}$

When we dehomogenise U with respect to y, we obtain the set $d(U) := \{x_1^2,\ x_1\}$ with multiplicative variables as follows.

$d(u)$	$\mathcal{M}_{\mathcal{J}}(d(u), d(U))$
x_1^2	$\{x_1\}$
x_1	$\emptyset$

It is now clear that Janet is not an extendible involutive division with respect to Lex or DegRevLex, as $\mathcal{M}_{\mathcal{J}}(x_1y^2, U) \setminus \{y\} \neq \mathcal{M}_{\mathcal{J}}(x_1, d(U))$, or $\{x_1\} \neq \emptyset$. □

Proposition 4.5.9 *Let G' be a set of polynomials over a polynomial ring containing variables $x_1, \ldots, x_n$ and a homogenising variable y. If (i) G' is an Involutive Basis with respect to some extendible monomial ordering O and some involutive division I that is extendible with respect to O; and (ii) $d(G')$ is an autoreduced set, then $d(G')$ is an Involutive Basis with respect to O and I.*

Proof: By Definition 4.2.7, we can show that $d(G')$ is an Involutive Basis with respect to O and I by showing that any multiple $d(g')t$ of any polynomial $d(g') \in d(G')$ by any term t involutively reduces to zero using $d(G')$. Because G' is an Involutive Basis with respect to O and I, the polynomial $g't$ involutively reduces to zero using G' by the series of involutive reductions

$$g't \xrightarrow[I\ g'_{\alpha_1}]{} h_1 \xrightarrow[I\ g'_{\alpha_2}]{} h_2 \xrightarrow[I\ g'_{\alpha_3}]{} \cdots \xrightarrow[I\ g'_{\alpha_A}]{} 0,$$

where $g'_{\alpha_i} \in G'$ for all $1 \leqslant i \leqslant A$.

Claim: The polynomial $d(g')t$ involutively reduces to zero using $d(G')$ by the series of involutive reductions

$$d(g')t \xrightarrow[I\ d(g'_{\alpha_1})]{} d(h_1) \xrightarrow[I\ d(g'_{\alpha_2})]{} d(h_2) \xrightarrow[I\ d(g'_{\alpha_3})]{} \cdots \xrightarrow[I\ d(g'_{\alpha_A})]{} 0,$$

where $d(g'_{\alpha_i}) \in d(G')$ for all $1 \leqslant i \leqslant A$.

Proof of Claim: It is clear that if a polynomial $g'_j \in G'$ involutively reduces a polynomial h, then the polynomial $d(g'_j) \in d(G')$ will always conventionally reduce the polynomial $d(h)$. Further, knowing that I is extendible with respect to O, we can state that $d(g'_j)$ will also always involutively reduce $d(h)$. The result now follows by noticing that $d(G')$ is autoreduced, so that $d(g'_j)$ is the only possible involutive divisor of $d(h)$, and hence the above series of involutive reductions is the only possible way of involutively reducing the polynomial $d(g')t$. $\square$

Open Question 1 If the set G returned by the procedure outlined in Definition 4.5.1 is not autoreduced, under what circumstances does autoreducing G result in obtaining a set that is an Involutive Basis for the ideal generated by F?

Let us now consider two examples illustrating that the set G returned by the procedure outlined in Definition 4.5.1 may or may not be autoreduced.

Example 4.5.10 Let $F := \{2x_1x_2 + x_1^2 + 5,\ x_2^2 + x_1 + 8\}$ be a basis generating an ideal J over the polynomial ring $\mathbb{Q}[x_1, x_2]$, and let the monomial ordering be InvLex. Ordinarily, we can compute an Involutive Basis $G := \{x_2^2 + x_1 + 8,\ 2x_1x_2 + x_1^2 + 5,\ 10x_2 - x_1^3 - 4x_1^2 - 37x_1,\ x_1^4 + 4x_1^3 + 42x_1^2 + 25\}$ for F with respect to the Janet involutive division by using Algorithm 9.

Taking the homogeneous route (using Definition 4.5.1), we can homogenise F to obtain the basis $F' := \{2x_1x_2 + x_1^2 + 5y^2,\ x_2^2 + yx_1 + 8y^2\}$ over the polynomial ring $\mathbb{Q}[y, x_1, x_2]$, where y is the homogenising variable (which must be greater than all other variables). Computing an Involutive Basis for the set F' with respect to the Janet involutive division using Algorithm 9, we obtain the basis $G' := \{x_2^2+yx_1+8y^2,\ 2x_1x_2+x_1^2+5y^2,\ 10y^2x_2-x_1^3-4yx_1^2 - 37y^2x_1,\ x_1^4 + 4yx_1^3 + 42y^2x_1^2 + 25y^4\}$. When we dehomogenise this basis, we obtain the set $d(G') := \{x_2^2+x_1+8,\ 2x_1x_2+x_1^2+5,\ 10x_2-x_1^3-4x_1^2-37x_1,\ x_1^4+4x_1^3+42x_1^2+25\}$.

It is now clear that the set $d(G')$ is autoreduced (and hence $d(G')$ is an Involutive Basis for J) because $d(G') = G$.

Example 4.5.11 Let $F := \{x_2^2 + 2x_1x_2 + 5,\ x_2 + x_1^2 + 8\}$ be a basis generating an ideal J over the polynomial ring $\mathbb{Q}[x_1, x_2]$, and let the monomial ordering be InvLex. Ordinarily, we can compute an Involutive Basis $G := \{x_2^2 - 2x_1^3 - 16x_1 + 5,\ x_2 + x_1^2 + 8,\ x_1^4 - 2x_1^3 + 16x_1^2 - 16x_1 + 69\}$ for F with respect to the Janet involutive division by using Algorithm 9.

Taking the homogeneous route (using Definition 4.5.1), we can homogenise F to obtain the basis $F' := \{x_2^2 + 2x_1x_2 + 5y^2,\ yx_2 + x_1^2 + 8y^2\}$ over the polynomial ring $\mathbb{Q}[y, x_1, x_2]$, where y is the homogenising variable (which must be greater than all other variables). Computing an Involutive Basis for the set F' with respect to the Janet involutive division using Algorithm 9, we obtain the basis $G' := \{x_2^2 + 2x_1x_2 + 5y^2,\ x_1^2x_2 + 2x_1^3 - 8yx_1^2 + 16y^2x_1 - 69y^3,\ yx_1x_2 + x_1^3 + 8y^2x_1,\ yx_2 + x_1^2 + 8y^2,\ x_1^4 - 2yx_1^3 + 16y^2x_1^2 - 16y^3x_1 + 69y^4\}$. When we dehomogenise this basis, we obtain the set $d(G') := \{x_2^2 + 2x_1x_2 + 5,\ x_1^2x_2 + 2x_1^3 - 8x_1^2 + 16x_1 - 69,\ x_1x_2 + x_1^3 + 8x_1,\ x_2 + x_1^2 + 8,\ x_1^4 - 2x_1^3 + 16x_1^2 - 16x_1 + 69\}$. This time however, because the set $d(G')$ is not autoreduced (the polynomial $x_1x_2 + x_1^3 + 8x_1 \in d(G')$ can involutively reduce the second term of the polynomial $x_2^2 + 2x_1x_2 + 5 \in d(G')$), we cannot deduce that $d(G')$ is an Involutive Basis for J.

Remark 4.5.12 Although the set G returned by the procedure outlined in Definition 4.5.1 may not always be an Involutive Basis for the ideal generated by F, because the set G' will always be an Involutive Basis (and hence also a Gröbner Basis), we can state that G will always be a Gröbner Basis for the ideal generated by F (cf. Definition 2.5.7).

4.5.3 Logged Involutive Bases

Just as a Logged Gröbner Basis expresses each member of the Gröbner Basis in terms of members of the original basis from which the Gröbner Basis was computed, a Logged Involutive Basis expresses each member of the Involutive Basis in terms of members of the original basis from which the Involutive Basis was computed.

Definition 4.5.13 Let $G = \{g_1, \dots, g_p\}$ be an Involutive Basis computed from an initial basis $F = \{f_1, \dots, f_m\}$. We say that G is a *Logged Involutive Basis* if, for each $g_i \in G$,

we have an explicit expression of the form

$$g_i = \sum_{\alpha=1}^{\beta} t_\alpha f_{k_\alpha},$$

where the t_α are terms and $f_{k_\alpha} \in F$ for all $1 \leqslant \alpha \leqslant \beta$.

Proposition 4.5.14 *Given a finite basis $F = \{f_1, \ldots, f_m\}$, it is always possible to compute a Logged Involutive Basis for F.*

Proof: Let $G = \{g_1, \ldots, g_p\}$ be an Involutive Basis computed from the initial basis $F = \{f_1, \ldots, f_m\}$ using Algorithm 9 (where $f_i \in R[x_1, \ldots, x_n]$ for all $f_i \in F$). If an arbitrary $g_i \in G$ is not a member of the original basis F, then either g_i is an involutively reduced prolongation, or g_i is obtained through the process of autoreduction. In the former case, we can express g_i in terms of members of F by substitution because

$$g_i = hx_j - \sum_{\alpha=1}^{\beta} t_\alpha h_{k_\alpha}$$

for a variable x_j; terms t_α and polynomials h and h_{k_α} which we already know how to express in terms of members of F. In the latter case,

$$g_i = h - \sum_{\alpha=1}^{\beta} t_\alpha h_{k_\alpha}$$

for terms t_α and polynomials h and h_{k_α} which we already know how to express in terms of members of F, so it follows that we can again express g_i in terms of members of F. □

Chapter 5

Noncommutative Involutive Bases

In the previous chapter, we introduced the theory of commutative Involutive Bases and saw that such bases are always commutative Gröbner Bases with extra structure. In this chapter, we will follow a similar path, in that we will define an algorithm to compute a *noncommutative Involutive Basis* that will serve as an alternative method of obtaining a noncommutative Gröbner Basis, and the noncommutative Gröbner Bases we will obtain will also have some extra structure.

As illustrated by the diagram shown below, the theory of noncommutative Involutive Bases will draw upon all the theory that has come before in this thesis, and as a consequence will inherit many of the restrictions imposed by this theory. For example, our noncommutative Involutive Basis algorithm will not be guaranteed to terminate precisely because we are working in a noncommutative setting, and noncommutative involutive divisions will have properties that will influence the correctness and termination of the algorithm.

$$\begin{tikzcd}
\text{Commutative Gröbner Bases} \arrow[r] \arrow[d] & \text{Commutative Involutive Bases} \arrow[d] \\
\text{Noncommutative Gröbner Bases} \arrow[r] & \text{Noncommutative Involutive Bases}
\end{tikzcd}$$

5.1 Noncommutative Involutive Reduction

Recall that in a commutative polynomial ring, a monomial u_2 is an involutive divisor of a monomial u_1 if $u_1 = u_2u_3$ for some monomial u_3 and all variables in u_3 are multiplicative for u_2. In other words, we are able to form u_1 from u_2 by multiplying u_2 with multiplicative variables.

In a noncommutative polynomial ring, an involutive division will again induce a restricted form of division. However, because left and right multiplication are separate processes in noncommutative polynomial rings, we will require the notion of *left* and *right multiplicative* variables in order to determine whether a conventional divisor is an involutive divisor, so that (intuitively) a monomial u_2 will involutively divide a monomial u_1 if we are able to form u_1 from u_2 by multiplying u_2 on the left with left multiplicative variables and on the right by right multiplicative variables.

More formally, let u_1 and u_2 be two monomials over a noncommutative polynomial ring, and assume that u_1 is a conventional divisor of u_2, so that $u_1 = u_3u_2u_4$ for some monomials u_3 and u_4. Assume that an arbitrary noncommutative involutive division I partitions the variables in the polynomial ring into sets of *left multiplicative* and *left nonmultiplicative* variables for u_2, and also partitions the variables in the polynomial ring into sets of *right multiplicative* and *right nonmultiplicative* variables for u_2. Let us now define two methods of deciding whether u_2 is an *involutive* divisor of u_1 (written $u_2 \mid_I u_1$), the first of which will depend only on the *first* variable we multiply u_2 with on the left and on the right in order to form u_1, and the second of which will depend on *all* the variables we multiply u_2 with in order to form u_1.

Definition 5.1.1 Let $u_1 = u_3u_2u_4$, and let I be defined as in the previous paragraph.

- **(Thin Divisor)** $u_2 \mid_I u_1$ if the variable $\text{Suffix}(u_3, 1)$ (if it exists) is in the set of left multiplicative variables for u_2, and the variable $\text{Prefix}(u_4, 1)$ (again if it exists) is in the set of right multiplicative variables for u_2.
- **(Thick Divisor)** $u_2 \mid_I u_1$ if all the variables in u_3 are in the set of left multiplicative variables for u_2, and all the variables in u_4 are in the set of right multiplicative variables for u_2.

Remark 5.1.2 We introduce two methods for determining whether a conventional divisor is an involutive divisor because each of the methods has its own advantages and

disadvantages. From a theoretical standpoint, using thin divisors enables us to follow the path laid down in Chapter 4, in that we are able to show that a Locally Involutive Basis is an Involutive Basis by proving that the involutive division used is continuous, something that we cannot do if thick divisors are being used. On the other hand, once we have obtained our Locally Involutive Basis, involutive reduction with respect to thick divisors is more efficient than it is with respect to thin divisors, as less work is required in order to determine whether a monomial is involutively divisible by a set of monomials. For these reasons, we will use thin divisors when presenting the theory in this chapter (hence the following definition), and will only use thick divisors when, by doing so, we are able to gain some advantage.

Remark 5.1.3 Unless otherwise stated, from now on we will use thin divisors to determine whether a conventional divisor is an involutive divisor.

Example 5.1.4 Let $u_1 := xyz^2x$; $u_1' := yz^2y$ and $u_2 := z^2$ be three monomials over the polynomial ring $\mathcal{R} = \mathbb{Q}\langle x, y, z\rangle$, and let an involutive division I partition the variables in $\mathcal{R}$ into the following sets of variables for the monomial u_2: left multiplicative $= \{x, y\}$; left nonmultiplicative $= \{z\}$; right multiplicative $= \{x, z\}$; right nonmultiplicative $= \{y\}$. It is true that u_2 conventionally divides both monomials u_1 and u_1', but u_2 only involutively divides monomial u_1 as, defining $u_3 := xy$; $u_4 := x$; $u_3' = y$ and $u_4' = y$ (so that $u_1 = u_3u_2u_4$ and $u_1' = u_3'u_2u_4'$), we observe that the variable $\text{Suffix}(u_3, 1) = y$ is in the set of left multiplicative variables for u_2; the variable $\text{Prefix}(u_4, 1) = x$ is in the set of right multiplicative variables for u_2; but the variable $\text{Prefix}(u_4', 1) = y$ is not in the set of right multiplicative variables for u_2.

Let us now formally define what is meant by a (noncommutative) involutive division.

Definition 5.1.5 Let M denote the set of all monomials in a noncommutative polynomial ring $\mathcal{R} = R\langle x_1, \ldots, x_n\rangle$, and let $U \subset M$. The involutive cone $\mathcal{C}_I(u, U)$ of any monomial $u \in U$ with respect to some involutive division I is defined as follows.

$$\mathcal{C}_I(u, U) = \{v_1uv_2 \text{ such that } v_1, v_2 \in M \text{ and } u \mid_I v_1uv_2\}.$$

Definition 5.1.6 Let M denote the set of all monomials in a noncommutative polynomial ring $\mathcal{R} = R\langle x_1, \ldots, x_n\rangle$. A *strong* involutive division I is defined on M if, given any finite set of monomials $U \subset M$, we can assign a set of left multiplicative variables $\mathcal{M}_I^L(u, U) \subseteq$

$\{x_1, \ldots, x_n\}$ and a set of right multiplicative variables $\mathcal{M}_I^R(u, U) \subseteq \{x_1, \ldots, x_n\}$ to any monomial $u \in U$ such that the following three conditions are satisfied.

- If there exist two elements $u_1, u_2 \in U$ such that $\mathcal{C}_I(u_1, U) \cap \mathcal{C}_I(u_2, U) \neq \emptyset$, then either $\mathcal{C}_I(u_1, U) \subset \mathcal{C}_I(u_2, U)$ or $\mathcal{C}_I(u_2, U) \subset \mathcal{C}_I(u_1, U)$.
- Any monomial $v \in \mathcal{C}_I(u, U)$ is involutively divisible by u in one way only, so that if u appears as a subword of v in more than one way, then only one of these ways allows us to deduce that u is an involutive divisor of v.
- If $V \subset U$, then $\mathcal{M}_I^L(v, U) \subseteq \mathcal{M}_I^L(v, V)$ and $\mathcal{M}_I^R(v, U) \subseteq \mathcal{M}_I^R(v, V)$ for all $v \in V$.

If any of the above conditions are not satisfied, the involutive division is called a *weak* involutive division.

Remark 5.1.7 We shall refer to the three conditions of Definition 5.1.6 as (respectively) the Disjoint Cones condition, the Unique Divisor condition and the Subset condition.

Definition 5.1.8 Given an involutive division I, the involutive span $\mathcal{C}_I(U)$ of a set of noncommutative monomials U with respect to I is given by the expression

$$\mathcal{C}_I(U) = \bigcup_{u \in U} \mathcal{C}_I(u, U).$$

Remark 5.1.9 The (conventional) span of a set of noncommutative monomials U is given by the expression

$$\mathcal{C}(U) = \bigcup_{u \in U} \mathcal{C}(u, U),$$

where $\mathcal{C}(u, U) = \{v_1 u v_2 \text{ such that } v_1, v_2 \text{ are monomials}\}$ is the (conventional) cone of a monomial $u \in U$.

Definition 5.1.10 If an involutive division I determines the left and right multiplicative variables for a monomial $u \in U$ independent of the set U, then I is a *global* division. Otherwise, I is a *local* division.

Remark 5.1.11 The multiplicative variables for a set of polynomials P (whose terms are ordered by a monomial ordering O) are determined by the multiplicative variables for the set of leading monomials $\mathrm{LM}(P)$.

In Algorithm 10, we specify how to involutively divide a polynomial p with respect to a set of polynomials P using thin divisors. Note that this algorithm combines the modifications made to Algorithm 1 in Algorithms 2 and 7.

Algorithm 10 The Noncommutative Involutive Division Algorithm

Input: A nonzero polynomial p and a set of nonzero polynomials $P = \{p_1, \ldots, p_m\}$ over a polynomial ring $R\langle x_1, \ldots x_n\rangle$; an admissible monomial ordering O; an involutive division I.

Output: $\mathrm{Rem}_I(p, P) := r$, the involutive remainder of p with respect to P.

$r = 0$;
while $(p \neq 0)$ **do**
  $u = \mathrm{LM}(p)$; $c = \mathrm{LC}(p)$; $j = 1$; found = false;
  while $(j \leqslant m)$ **and** (found == false) **do**
    if $(\mathrm{LM}(p_j) \mid_I u)$ **then**
      found = true;
      choose u_ℓ and u_r such that $u = u_\ell \mathrm{LM}(p_j) u_r$, the variable $\mathrm{Suffix}(u_\ell, 1)$ (if it exists) is left multiplicative for p_j, and the variable $\mathrm{Prefix}(u_r, 1)$ (again if it exists) is right multiplicative for p_j;
      $p = p - (c\mathrm{LC}(p_j)^{-1}) u_\ell p_j u_r$;
    else
      $j = j + 1$;
    end if
  end while
  if (found == false) **then**
    $r = r + \mathrm{LT}(p)$; $p = p - \mathrm{LT}(p)$;
  end if
end while
return r;

Remark 5.1.12 Continuing the convention from Algorithm 2, we will always choose the u_ℓ with the smallest degree in the line 'choose u_ℓ and u_r such that...' in Algorithm 10.

Example 5.1.13 Let $P := \{x^2 - 2y,\ xy - x,\ y^3 + 3\}$ be a set of polynomials over the polynomial ring $\mathbb{Q}\langle x, y\rangle$ ordered with respect to the DegLex monomial ordering, and assume that an involutive division I assigns multiplicative variables to P as follows.

p	$\mathcal{M}_I^L(\mathrm{LM}(p), \mathrm{LM}(P))$	$\mathcal{M}_I^R(\mathrm{LM}(p), \mathrm{LM}(P))$
$x^2 - 2y$	$\{x, y\}$	$\{x\}$
$xy - x$	$\{y\}$	$\{x, y\}$
$y^3 + 3$	$\{x\}$	$\emptyset$

Here is a dry run for Algorithm 10 when we involutively divide the polynomial $p := 2x^2y^3 + yxy$ with respect to P to obtain the polynomial $yx - 12y$, where A; B; C and D refer to the tests $(p \neq 0)$?; $((j \leqslant 3)$ and (found == false))?; $(\mathrm{LM}(p_j) \mid_I u)$? and (found == false)? respectively.

p	r	u	c	j	found	u_ℓ	u_r	A	B	C	D
$2x^2y^3 + yxy$	0							true			
		x^2y^3	2	1	false				true	false	
				2					true	false	
				3					true	true	
$yxy - 6x^2$					true	x^2	1		false		false
								true			
		yxy	1	1	false				true	false	
				2					true	true	
$-6x^2 + yx$					true	y	1		false		false
								true			
		x^2	-6	1	false				true	true	
$yx - 12y$					true	1	1		false		false
								true			
		yx	1	1	false				true	false	
				2					true	false	
				3					true	false	
				4					false		true
$-12y$	yx							true			
		y	-12	1	false				true	false	
				2					true	false	
				3					true	false	
				4					false		true
0	$yx - 12y$							false			

5.2 Prolongations and Autoreduction

Just as in the commutative case, we will compute a (noncommutative) Locally Involutive Basis by using *prolongations* and *autoreduction*, but here we have to distinguish between *left prolongations* and *right prolongations*.

Definition 5.2.1 Given a set of polynomials P, a *left prolongation* of a polynomial $p \in P$ is a product $x_i p$, where $x_i \notin \mathcal{M}_I^L(\mathrm{LM}(p), \mathrm{LM}(P))$ with respect to some involutive division I; and a *right prolongation* of a polynomial $p \in P$ is a product px_i, where $x_i \notin \mathcal{M}_I^R(\mathrm{LM}(p), \mathrm{LM}(P))$ with respect to some involutive division I.

Definition 5.2.2 A set of polynomials P is said to be *autoreduced* if no polynomial $p \in P$ exists such that p contains a term which is involutively divisible (with respect to P) by some polynomial $p' \in P \setminus \{p\}$.

Algorithm 11 The Noncommutative Autoreduction Algorithm

Input: A set of polynomials $P = \{p_1, p_2, \ldots, p_\alpha\}$; an involutive division I.

Output: An autoreduced set of polynomials $Q = \{q_1, q_2, \ldots, q_\beta\}$.

while ($\exists\ p_i \in P$ such that $\mathrm{Rem}_I(p_i, P, P \setminus \{p_i\}) \neq p_i$) **do**
  $p_i' = \mathrm{Rem}_I(p_i, P, P \setminus \{p_i\})$;
  $P = P \setminus \{p_i\}$;
  if ($p_i' \neq 0$) **then**
    $P = P \cup \{p_i'\}$;
  end if
end while
$Q = P$;
return Q;

Remark 5.2.3 With respect to a strong involutive division, the involutive cones of an autoreduced set of polynomials are always disjoint.

Remark 5.2.4 The notation $\mathrm{Rem}_I(p_i, P, P \setminus \{p_i\})$ used in Algorithm 11 has the same meaning as in Definition 4.2.2.

Proposition 5.2.5 *Let P be a set of polynomials over a noncommutative polynomial ring $\mathcal{R} = R\langle x_1, \ldots, x_n\rangle$, and let f and g be two polynomials also in $\mathcal{R}$. If P is autoreduced with respect to a strong involutive division I, then* $\mathrm{Rem}_I(f, P) + \mathrm{Rem}_I(g, P) = \mathrm{Rem}_I(f + g, P)$.

Proof: Let $f' := \mathrm{Rem}_I(f, P)$; $g' := \mathrm{Rem}_I(g, P)$ and $h' := \mathrm{Rem}_I(h, P)$, where $h := f + g$. Then, by the respective involutive reductions, we have expressions

$$f' = f - \sum_{a=1}^{A} u_a p_{\alpha_a} v_a;$$

$$g' = g - \sum_{b=1}^{B} u_b p_{\beta_b} v_b$$

and

$$h' = h - \sum_{c=1}^{C} u_c p_{\gamma_c} v_c,$$

where p_{α_a}, p_{β_b}, $p_{\gamma_c} \in P$ and $u_a, v_a, u_b, v_b, u_c, v_c$ are terms such that each p_{α_a}, p_{β_b} and p_{γ_c} involutively divides each $u_a p_{\alpha_a} v_a$, $u_b p_{\beta_b} v_b$ and $u_c p_{\gamma_c} v_c$ respectively.

Consider the polynomial $h' - f' - g'$. By the above expressions, we can deduce[1] that

$$h' - f' - g' = \sum_{a=1}^{A} u_a p_{\alpha_a} v_a + \sum_{b=1}^{B} u_b p_{\beta_b} v_b - \sum_{c=1}^{C} u_c p_{\gamma_c} v_c =: \sum_{d=1}^{D} u_d p_{\delta_d} v_d.$$

Claim: $\mathrm{Rem}_I(h' - f' - g', P) = 0$.

Proof of Claim: Let t denote the leading term of the polynomial $\sum_{d=1}^{D} u_d p_{\delta_d} v_d$. Then $\mathrm{LM}(t) = \mathrm{LM}(u_k p_{\delta_k} v_k)$ for some $1 \leqslant k \leqslant D$ since, if not, there exists a monomial

$$\mathrm{LM}(u_{k'} p_{\delta_{k'}} v_{k'}) = \mathrm{LM}(u_{k''} p_{\delta_{k''}} v_{k''}) =: w$$

for some $1 \leqslant k', k'' \leqslant D$ (with $p_{\delta_{k'}} \neq p_{\delta_{k''}}$) such that w is involutively divisible by the two polynomials $p_{\delta_{k'}}$ and $p_{\delta_{k''}}$, contradicting Definition 5.1.6 (recall that I is strong and P is autoreduced, so that the involutive cones of P are disjoint). It follows that we can use p_{δ_k} to eliminate t by involutively reducing $h' - f' - g'$ as shown below.

$$\sum_{d=1}^{D} u_d p_{\delta_d} v_d \xrightarrow[I]{\ p_{\delta_k}} \sum_{d=1}^{k-1} u_d p_{\delta_d} v_d + \sum_{d=k+1}^{D} u_d p_{\delta_d} v_d. \tag{5.1}$$

By induction, we can apply a chain of involutive reductions to the right hand side of Equation (5.1) to obtain a zero remainder, so that $\mathrm{Rem}_I(h' - f' - g', P) = 0$. $\square$

[1] For $1 \leqslant d \leqslant A$, $u_d p_{\delta_d} v_d = u_a p_{\alpha_a} v_a$ $(1 \leqslant a \leqslant A)$; for $A+1 \leqslant d \leqslant A+B$, $u_d p_{\delta_d} v_d = u_b p_{\beta_b} v_b$ $(1 \leqslant b \leqslant B)$; and for $A+B+1 \leqslant d \leqslant A+B+C =: D$, $u_d p_{\delta_d} v_d = u_c p_{\gamma_c} v_c$ $(1 \leqslant c \leqslant C)$.

To complete the proof, we note that since f', g' and h' are all involutively irreducible, we must have $\mathrm{Rem}_I(h' - f' - g', P) = h' - f' - g'$. It therefore follows that $h' - f' - g' = 0$, or $h' = f' + g'$ as required. □

Definition 5.2.6 Given an involutive division I and an admissible monomial ordering O, an autoreduced set of noncommutative polynomials P is a *Locally Involutive Basis* with respect to I and O if any (left or right) prolongation of any polynomial $p_i \in P$ involutively reduces to zero using P.

Definition 5.2.7 Given an involutive division I and an admissible monomial ordering O, an autoreduced set of noncommutative polynomials P is an *Involutive Basis* with respect to I and O if any multiple up_iv of any polynomial $p_i \in P$ by any terms u and v involutively reduces to zero using P.

5.3 The Noncommutative Involutive Basis Algorithm

To compute a (noncommutative) Locally Involutive Basis, we use Algorithm 12, an algorithm that is virtually identical to Algorithm 9, apart from the fact that at the beginning of the first **while** loop, the set S is constructed in different ways.

5.4 Continuity and Conclusivity

In the commutative case, when we construct a Locally Involutive Basis using Algorithm 9, we know that the algorithm will always return a commutative Gröbner Basis as long as we use an admissible monomial ordering and the chosen involutive division possesses certain properties. In summary,

(a) Any Locally Involutive Basis returned by Algorithm 9 is an Involutive Basis if the involutive division used is continuous (Proposition 4.3.3);

(b) Algorithm 9 always terminates if (in addition) the involutive division used is constructive, Noetherian and stable (Proposition 4.4.8);

(c) Every Involutive Basis is a Gröbner Basis (Theorem 4.4.9).

In the noncommutative case, we cannot hope to produce a carbon copy of the above results because a finitely generated basis may have an infinite Gröbner Basis, leading to

Algorithm 12 The Noncommutative Involutive Basis Algorithm

Input: A Basis $F = \{f_1, f_2, \ldots, f_m\}$ for an ideal J over a noncommutative polynomial ring $R\langle x_1, \ldots x_n\rangle$; an admissible monomial ordering O; an involutive division I.

Output: A Locally Involutive Basis $G = \{g_1, g_2, \ldots, g_p\}$ for J (in the case of termination).

$G = \emptyset$;
$F = \text{Autoreduce}(F)$;
while $(G == \emptyset)$ **do**
  $S = \{x_i f \mid f \in F,\ x_i \notin \mathcal{M}_I^L(f, F)\} \cup \{f x_i \mid f \in F,\ x_i \notin \mathcal{M}_I^R(f, F)\}$;
  $s' = 0$;
  while $(S \neq \emptyset)$ **and** $(s' == 0)$ **do**
    Let s be a polynomial in S whose lead monomial is minimal with respect to O;
    $S = S \setminus \{s\}$;
    $s' = \text{Rem}_I(s, F)$;
  end while
  if $(s' \neq 0)$ **then**
    $F = \text{Autoreduce}(F \cup \{s'\})$;
  else
    $G = F$;
  end if
end while
return G;

the conclusion that Algorithm 12 does not always terminate. The best we can therefore hope for is if an ideal generated by a set of polynomials F possesses a finite Gröbner Basis with respect to some admissible monomial ordering O, then F also possesses a finite Involutive Basis with respect to O and some involutive division I. We shall call any involutive division that possesses this property *conclusive*.

Definition 5.4.1 Let F be an arbitrary basis generating an ideal over a noncommutative polynomial ring, and let O be an arbitrary admissible monomial ordering. An involutive division I is *conclusive* if Algorithm 12 terminates with F, I and O as input whenever Algorithm 5 terminates with F and O as input.

Of course it is easy enough to define the above property, but much harder to prove that a particular involutive division is conclusive. In fact, no involutive division defined in this thesis will be shown to be conclusive, and the existence of such divisions will be left as an open question.

5.4.1 Properties for Strong Involutive Divisions

Here is a summary of facts that can be deduced when using a strong involutive division.

(a) Any Locally Involutive Basis returned by Algorithm 12 is an Involutive Basis if the involutive division used is strong and continuous (Proposition 5.4.3);

(b) Algorithm 12 always terminates whenever Algorithm 5 terminates if (in addition) the involutive division used is conclusive;

(c) Every Involutive Basis with respect to a strong involutive division is a Gröbner Basis (Theorem 5.4.4).

Let us now prove the assertions made in parts (a) and (c) of the above list, beginning by defining what is meant by a continuous involutive division in the noncommutative case.

Definition 5.4.2 Let I be a fixed involutive division; let w be a fixed monomial; let U be any set of monomials; and consider any sequence $(u_1,\ u_2,\ \ldots,\ u_k)$ of monomials from U ($u_i \in U$ for all $1 \leqslant i \leqslant k$), each of which is a conventional divisor of w (so that $w = \ell_i u_i r_i$ for all $1 \leqslant i \leqslant k$, where the ℓ_i and the r_i are monomials). For all $1 \leqslant i < k$, suppose that the monomial u_{i+1} satisfies exactly one of the following conditions.

(a) u_{i+1} involutively divides a left prolongation of u_i, so that $\deg(\ell_i) \geqslant 1$; $\text{Suffix}(\ell_i, 1) \notin \mathcal{M}_I^L(u_i, U)$; and $u_{i+1} \mid_I (\text{Suffix}(\ell_i, 1))u_i$.

(b) u_{i+1} involutively divides a right prolongation of u_i, so that $\deg(r_i) \geqslant 1$; $\text{Prefix}(r_i, 1) \notin \mathcal{M}_I^R(u_i, U)$; and $u_{i+1} \mid_I u_i(\text{Prefix}(r_i, 1))$.

Then I is *continuous at* w if all the pairs (ℓ_i, r_i) are distinct ($(\ell_i, r_i) \neq (\ell_j, r_j)$ for all $i \neq j$); I is a *continuous* involutive division if I is continuous for all possible w.

Proposition 5.4.3 *If an involutive division I is strong and continuous, and a given set of polynomials P is a Locally Involutive Basis with respect to I and some admissible monomial ordering O, then P is an Involutive Basis with respect to I and O.*

Proof: Let I be a strong and continuous involutive division; let O be an admissible monomial ordering; and let P be a Locally Involutive Basis with respect to I and O. Given any polynomial $p \in P$ and any terms u and v, in order to show that P is an Involutive Basis with respect to I and O, we must show that $upv \xrightarrow[I]{} {}_P\, 0$.

If $p \mid_I upv$ we are done, as we can use p to involutively reduce upv to obtain a zero remainder. Otherwise, either $\exists\, y_1 \notin \mathcal{M}_I^L(\text{LM}(p), \text{LM}(P))$ such that $y_1 = \text{Suffix}(u, 1)$, or $\exists\, y_1 \notin \mathcal{M}_I^R(\text{LM}(p), \text{LM}(P))$ such that $y_1 = \text{Prefix}(v, 1)$. Without loss of generality, assume that the first case applies. By Local Involutivity, the prolongation $y_1 p$ involutively reduces to zero using P. Assuming that the first step of this involutive reduction involves the polynomial $p_1 \in P$, we can write

$$y_1 p = u_1 p_1 v_1 + \sum_{a=1}^{A} u_{\alpha_a} p_{\alpha_a} v_{\alpha_a}, \tag{5.2}$$

where $p_{\alpha_a} \in P$ and $u_1, v_1, u_{\alpha_a}, v_{\alpha_a}$ are terms such that p_1 and each p_{α_a} involutively divide $u_1 p_1 v_1$ and each $u_{\alpha_a} p_{\alpha_a} v_{\alpha_a}$ respectively. Multiplying both sides of Equation (5.2) on the left by $u' := \text{Prefix}(u, \deg(u) - 1)$ and on the right by v, we obtain the equation

$$upv = u' u_1 p_1 v_1 v + \sum_{a=1}^{A} u' u_{\alpha_a} p_{\alpha_a} v_{\alpha_a} v. \tag{5.3}$$

If $p_1 \mid_I upv$, it is clear that we can use p_1 to involutively reduce the polynomial upv to obtain the polynomial $\sum_{a=1}^{A} u' u_{\alpha_a} p_{\alpha_a} v_{\alpha_a} v$. By Proposition 5.2.5, we can then continue

to involutively reduce upv by repeating this proof on each polynomial $u'u_{\alpha_a}p_{\alpha_a}v_{\alpha_a}v$ individually (where $1 \leqslant a \leqslant A$), noting that this process will terminate because of the admissibility of O (we have $\mathrm{LM}(u'u_{\alpha_a}p_{\alpha_a}v_{\alpha_a}v) < \mathrm{LM}(upv)$ for all $1 \leqslant a \leqslant A$).

Otherwise, if p_1 does not involutively divide upv, either $\exists\, y_2 \notin \mathcal{M}_I^L(\mathrm{LM}(p_1), \mathrm{LM}(P))$ such that $y_2 = \mathrm{Suffix}(u'u_1, 1)$, or $\exists\, y_2 \notin \mathcal{M}_I^R(\mathrm{LM}(p_1), \mathrm{LM}(P))$ such that $y_2 = \mathrm{Prefix}(v_1v, 1)$. This time (again without loss of generality), assume that the second case applies. By Local Involutivity, the prolongation p_1y_2 involutively reduces to zero using P. Assuming that the first step of this involutive reduction involves the polynomial $p_2 \in P$, we can write

$$p_1y_2 = u_2p_2v_2 + \sum_{b=1}^{B} u_{\beta_b}p_{\beta_b}v_{\beta_b}, \tag{5.4}$$

where $p_{\beta_b} \in P$ and $u_2, v_2, u_{\beta_b}, v_{\beta_b}$ are terms such that p_2 and each p_{β_b} involutively divide $u_2p_2v_2$ and each $u_{\beta_b}p_{\beta_b}v_{\beta_b}$ respectively. Multiplying both sides of Equation (5.4) on the left by $u'u_1$ and on the right by $v' := \mathrm{Suffix}(v_1v, \deg(v_1v) - 1)$, we obtain the equation

$$u'u_1p_1v_1v = u'u_1u_2p_2v_2v' + \sum_{b=1}^{B} u'u_1u_{\beta_b}p_{\beta_b}v_{\beta_b}v'. \tag{5.5}$$

Substituting for $u'u_1p_1v_1v$ from Equation (5.5) into Equation (5.3), we obtain the equation

$$upv = u'u_1u_2p_2v_2v' + \sum_{a=1}^{A} u'u_{\alpha_a}p_{\alpha_a}v_{\alpha_a}v + \sum_{b=1}^{B} u'u_1u_{\beta_b}p_{\beta_b}v_{\beta_b}v'. \tag{5.6}$$

If $p_2 \mid_I upv$, it is clear that we can use p_2 to involutively reduce the polynomial upv to obtain the polynomial $\sum_{a=1}^{A} u'u_{\alpha_a}p_{\alpha_a}v_{\alpha_a}v + \sum_{b=1}^{B} u'u_1u_{\beta_b}p_{\beta_b}v_{\beta_b}v'$. As before, we can then use Proposition 5.2.5 to continue the involutive reduction of upv by repeating this proof on each summand individually.

Otherwise, if p_2 does not involutively divide upv, we continue by induction, obtaining a sequence $p, p_1, p_2, p_3, \ldots$ of elements in P. By construction, each element in the sequence divides upv. By continuity (at $\mathrm{LM}(upv)$), no two elements in the sequence divide upv in the same way. Because upv has a finite number of subwords, the sequence must be finite, terminating with an involutive divisor $p' \in P$ of upv, which then allows us to finish the proof through use of Proposition 5.2.5 and the admissibility of O. □

Theorem 5.4.4 *An Involutive Basis with respect to a strong involutive division is a Gröbner Basis.*

Proof: Let $G = \{g_1, \ldots, g_m\}$ be an Involutive Basis with respect to some strong involutive division I and some admissible monomial ordering O, where each $g_i \in G$ (for all $1 \leqslant i \leqslant m$) is a member of the polynomial ring $R\langle x_1, \ldots, x_n\rangle$. To prove that G is a Gröbner Basis, we must show that all S-polynomials involving elements of G conventionally reduce to zero using G. Recall that each S-polynomial corresponds to an overlap between the lead monomials of two (not necessarily distinct) elements $g_i, g_j \in G$. Consider such an arbitrary overlap, with corresponding S-polynomial

$$\text{S-pol}(\ell_i, g_i, \ell_j, g_j) = c_2\ell_i g_i r_i - c_1\ell_j g_j r_j.$$

Because G is an Involutive Basis, it is clear that $c_2\ell_i g_i r_i \xrightarrow[I]{}_G 0$ and $c_1\ell_j g_j r_j \xrightarrow[I]{}_G 0$. By Proposition 5.2.5, it follows that $\text{S-pol}(\ell_i, g_i, \ell_j, g_j) \xrightarrow[I]{}_G 0$. But every involutive reduction is a conventional reduction, so we can deduce that $\text{S-pol}(\ell_i, g_i, \ell_j, g_j) \to_G 0$ as required. $\square$

Lemma 5.4.5 *Given an Involutive Basis G with respect to a strong involutive division, remainders are involutively unique with respect to G.*

Proof: Let G be an Involutive Basis with respect to some strong involutive division I and some admissible monomial ordering O. Theorem 5.4.4 tells us that G is a Gröbner Basis with respect to O and thus remainders are conventionally unique with respect to G. To prove that remainders are involutively unique with respect to G, we must show that the conventional and involutive remainders of an arbitrary polynomial p with respect to G are identical. For this it is sufficient to show that a polynomial p is conventionally reducible by G if and only if it is involutively reducible by G. ($\Rightarrow$) Trivial as every involutive reduction is a conventional reduction. ($\Leftarrow$) If a polynomial p is conventionally reducible by a polynomial $g \in G$, it follows that $\text{LM}(p) = u\text{LM}(g)v$ for some monomials u and v. But G is an Involutive Basis, so there must exist a polynomial $g' \in G$ such that $\text{LM}(g') \mid_I u\text{LM}(g)v$. Thus p is also involutively reducible by G. $\square$

5.4.2 Properties for Weak Involutive Divisions

While it is true that the previous three results (Proposition 5.4.3, Theorem 5.4.4 and Lemma 5.4.5) do not apply if a weak involutive division has been chosen, we will now show that corresponding results can be obtained for weak involutive divisions that are also *Gröbner* involutive divisions.

Definition 5.4.6 A weak involutive division I is a *Gröbner* involutive division if every Locally Involutive Basis with respect to I is a Gröbner Basis.

It is an easy consequence of Definition 5.4.6 that any Involutive Basis with respect to a weak and Gröbner involutive division is a Gröbner Basis; it therefore follows that we can also prove an analog of Lemma 5.4.5 for such divisions. To complete the mirroring of the results of Proposition 5.4.3, Theorem 5.4.4 and Lemma 5.4.5 for weak and Gröbner involutive divisions, it remains to show that a Locally Involutive Basis with respect to a weak; continuous and Gröbner involutive division is an Involutive Basis.

Proposition 5.4.7 *If an involutive division I is weak; continuous and Gröbner, and if a given set of polynomials P is a Locally Involutive Basis with respect to I and some admissible monomial ordering O, then P is an Involutive Basis with respect to I and O.*

Proof: Let I be a weak; continuous and Gröbner involutive division; let O be an admissible monomial ordering; and let P be a Locally Involutive Basis with respect to I and O. Given any polynomial $p \in P$ and any terms u and v, in order to show that P is an Involutive Basis with respect to I and O, we must show that $upv \xrightarrow[I]{} {}_P\, 0$.

For the first part of the proof, we proceed as in the proof of Proposition 5.4.3 to find an involutive divisor $p' \in P$ of upv using the continuity of I at $\mathrm{LM}(upv)$. This then allows us to involutive reduce upv using p' to obtain a polynomial q of the form

$$q = \sum_{a=1}^{A} u_a p_{\alpha_a} v_a, \tag{5.7}$$

where $p_{\alpha_a} \in P$ and the u_a and the v_a are terms.

For the second part of the proof, we now use the fact that P is a Gröbner Basis to find a polynomial $q' \in P$ such that q' conventionally divides q (such a polynomial will always exist because q is clearly a member of the ideal generated by P). If q' is an involutive divisor of q, then we can use q' to involutively reduce q to obtain a polynomial r of the form shown in Equation (5.7). Otherwise, if q' is not an involutive divisor of q, we can use the fact that I is continuous at $\mathrm{LM}(q)$ to find such an involutive divisor, which we can then use to involutive reduce q to obtain a polynomial r, again of the form shown in Equation (5.7). In both cases, we now proceed by induction on r, noting that this process will terminate because of the admissibility of O (we have $\mathrm{LM}(r) < \mathrm{LM}(q)$). $\square$

To summarise, here is the situation for weak and Gröbner involutive divisions.

(a) Any Locally Involutive Basis returned by Algorithm 12 is an Involutive Basis if the involutive division used is weak; continuous and Gröbner (Proposition 5.4.7);

(b) Algorithm 12 always terminates whenever Algorithm 5 terminates if (in addition) the involutive division used is conclusive;

(c) Every Involutive Basis with respect to a weak and Gröbner involutive division is a Gröbner Basis.

5.5 Noncommutative Involutive Divisions

Before we consider some examples of useful noncommutative involutive divisions, let us remark that it is possible to categorise any noncommutative involutive division somewhere between the following two *extreme* global divisions.

Definition 5.5.1 (The Empty Division) Given any monomial u, let u have no (left or right) multiplicative variables.

Definition 5.5.2 (The Full Division) Given any monomial u, let u have no (left or right) nonmultiplicative variables (in other words, all variables are left and right multiplicative for u).

Remark 5.5.3 It is clear that any set of polynomials G will be an Involutive Basis with respect to the (weak) full division as any multiple of a polynomial $g \in G$ will be involutively reducible by g (all conventional divisors are involutive divisors); in contrast it is impossible to find a finite Locally Involutive Basis for G with respect to the (strong) empty division as there will always be a prolongation of an element of the current basis that is involutively irreducible.

5.5.1 Two Global Divisions

Whereas most of the theory seen so far in this chapter has closely mirrored the corresponding commutative theory from Chapter 4, the commutative involutive divisions (Thomas, Janet and Pommaret) seen in the previous chapter do not generalise to the noncommutative case, or at the very least do not yield noncommutative involutive divisions of any

value. Despite this, an essential property of these divisions is that they ensure that the least common multiple $\text{lcm}(\text{LM}(p_1), \text{LM}(p_2))$ associated with an S-polynomial $\text{S-pol}(p_1, p_2)$ is involutively irreducible by at least one of p_1 and p_2, ensuring that the S-polynomial $\text{S-pol}(p_1, p_2)$ is constructed and involutively reduced during the course of the Involutive Basis algorithm.

To ensure that the corresponding process occurs in the noncommutative Involutive Basis algorithm, we must ensure that all overlap words associated to the S-polynomials of a particular basis are involutively irreducible (as placed in the overlap word) by at least one of the polynomials associated to each overlap word. This obviously holds true for the empty division, but it will also hold true for the following two global involutive divisions, where all variables are either assigned to be left multiplicative and right nonmultiplicative, or left nonmultiplicative and right multiplicative.

Definition 5.5.4 (The Left Division) Given any monomial u, the left division $\lhd$ assigns no left nonmultiplicative variables to u, and assigns no right multiplicative variables to u (in other words, all variables are left multiplicative and right nonmultiplicative for u).

Definition 5.5.5 (The Right Division) Given any monomial u, the right division $\rhd$ assigns no left multiplicative variables to u, and assigns no right nonmultiplicative variables to u (in other words, all variables are left nonmultiplicative and right multiplicative for u).

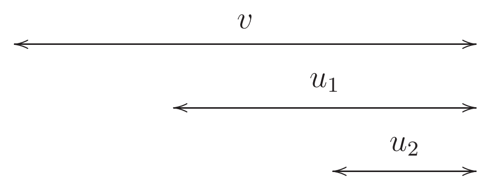
Proposition 5.5.6 *The left and right divisions are strong involutive divisions.*

Proof: We will only give the proof for the left division – the proof for the right division will follow by symmetry (replacing 'left' by 'right', and so on).

To prove that the left division is a strong involutive division, we need to show that the three conditions of Definition 5.1.6 hold.

- **Disjoint Cones Condition**
  Consider two involutive cones $\mathcal{C}_{\lhd}(u_1)$ and $\mathcal{C}_{\lhd}(u_2)$ associated to two monomials u_1, u_2 over some noncommutative polynomial ring $\mathcal{R}$. If $\mathcal{C}_{\lhd}(u_1) \cap \mathcal{C}_{\lhd}(u_2) \neq \emptyset$, then there must be some monomial $v \in \mathcal{R}$ such that v contains both monomials u_1 and u_2 as subwords, and (as placed in v) both u_1 and u_2 must be involutive divisors of v. By

definition of $\lhd$, both u_1 and u_2 must be suffices of v. Thus, assuming (without loss of generality) that $\deg(u_1) > \deg(u_2)$, we are able to draw the following diagram summarising the situation.



But now, assuming that $u_1 = u_3u_2$ for some monomial u_3, it is clear that $\mathcal{C}_{\lhd}(u_1) \subset \mathcal{C}_{\lhd}(u_2)$ because any monomial $w \in \mathcal{C}_{\lhd}(u_1)$ must be of the form $w = w'u_1$ for some monomial w'; this means that $w = w'u_3u_2 \in \mathcal{C}_{\lhd}(u_2)$.

- **Unique Divisor Condition**
  As a monomial v is only involutively divisible by a monomial u with respect to the left division if u is a suffix of v, it is clear that u can only involutively divide v in at most one way.

- **Subset Condition**
  Follows immediately due to the left division being a global division.

□

Proposition 5.5.7 *The left and right divisions are continuous.*

Proof: Again we will only treat the case of the left division. Let w be an arbitrary fixed monomial; let U be any set of monomials; and consider any sequence $(u_1,\ u_2,\ \ldots,\ u_k)$ of monomials from U ($u_i \in U$ for all $1 \leqslant i \leqslant k$), each of which is a conventional divisor of w (so that $w = \ell_i u_i r_i$ for all $1 \leqslant i \leqslant k$, where the ℓ_i and the r_i are monomials). For all $1 \leqslant i < k$, suppose that the monomial u_{i+1} satisfies condition (b) of Definition 5.4.2 (condition (a) can never be satisfied because $\lhd$ never assigns any left nonmultiplicative variables). To show that $\lhd$ is continuous, we must show that no two pairs (ℓ_i, r_i) and (ℓ_j, r_j) are the same, where $i \neq j$.

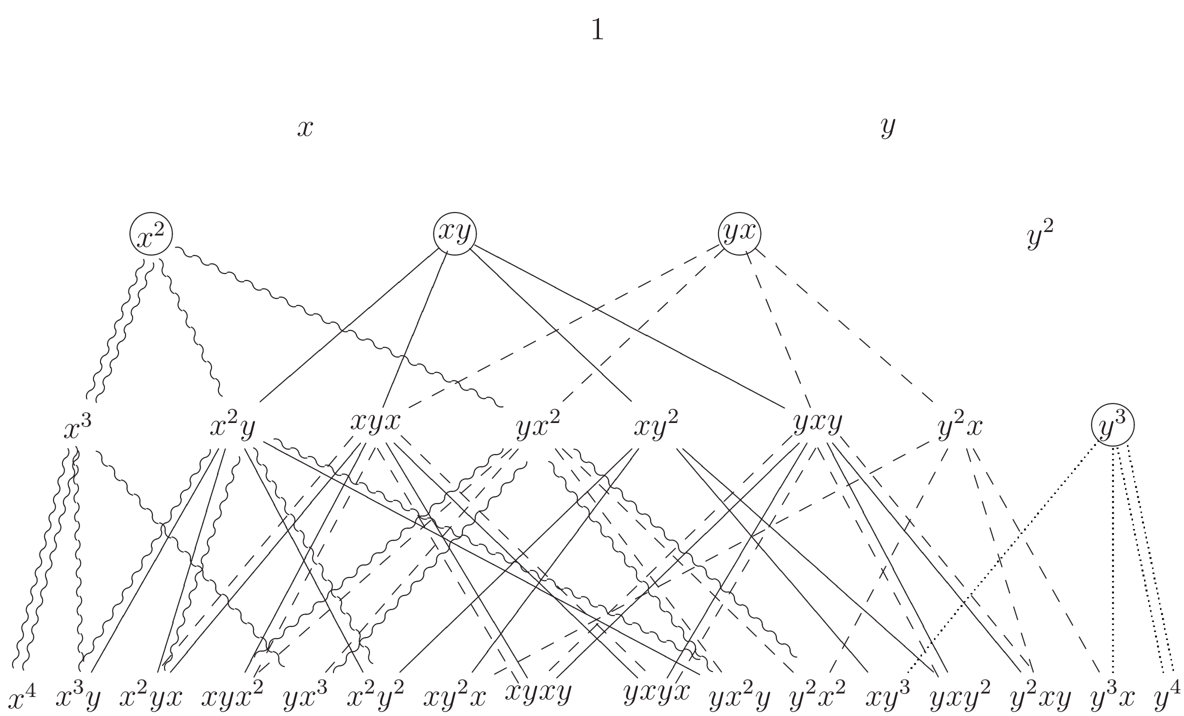
Consider an arbitrary monomial u_i from the sequence, where $1 \leqslant i < k$. Because $\lhd$ assigns no right multiplicative variables, the next monomial u_{i+1} in the sequence must be a suffix of the prolongation $u_i(\mathrm{Prefix}(r_i, 1))$ of u_i, so that $\deg(r_{i+1}) = \deg(r_i) - 1$.

It is therefore clear that no two identical (ℓ, r) pairs can be found in the sequence, as $\deg(r_1) > \deg(r_2) > \cdots > \deg(r_k)$. $\square$

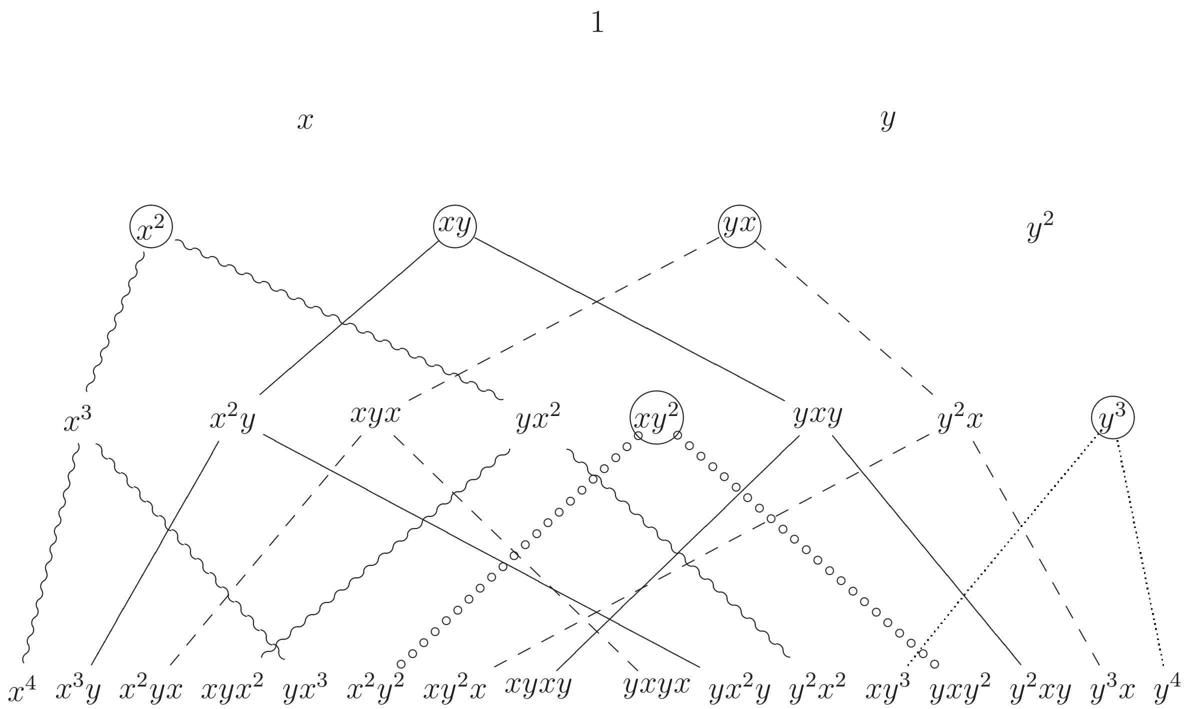
To illustrate the difference between the overlapping cones of a noncommutative Gröbner Basis and the disjoint cones of a noncommutative Involutive Basis with respect to the left division, consider the following example.

Example 5.5.8 Let $F := \{2xy + y^2 + 5,\ x^2 + y^2 + 8\}$ be a basis over the polynomial ring $\mathbb{Q}\langle x, y\rangle$, and let the monomial ordering be DegLex. Applying Algorithm 5 to F, we obtain the Gröbner Basis $G := \{2xy+y^2+5,\ x^2+y^2+8,\ 5y^3-10x+37y,\ 2yx+y^2+5\}$. Applying Algorithm 12 to F with respect to the left involutive division, we obtain the Involutive Basis $H := \{2xy + y^2 + 5,\ x^2 + y^2 + 8,\ 5y^3 - 10x + 37y,\ 5xy^2 + 5x - 6y,\ 2yx + y^2 + 5\}$.

To illustrate which monomials are reducible with respect to the Gröbner Basis, we can draw a monomial lattice, part of which is shown below. In the lattice, we draw a path from the (circled) lead monomial of any Gröbner Basis element to any multiple of that lead monomial, so that any monomial which lies on some path in the lattice is reducible by one or more Gröbner Basis elements. To distinguish between different Gröbner Basis elements we use different arrow types; we also arrange the lattice so that monomials of the same degree lie on the same level.



Notice that many of the monomials in the lattice are reducible by several of the Gröbner Basis elements. For example, the monomial x^2yx is reducible by the Gröbner Basis elements $2xy + y^2 + 5$; $x^2 + y^2 + 8$ and $2yx + y^2 + 5$. In contrast, any monomial in the corresponding lattice for the Involutive Basis may only be involutively reducible by at most one element in the Involutive Basis. We illustrate this by the following diagram, where we note that in the involutive lattice, a monomial only lies on a particular path if a member of the Involutive Basis is an involutive divisor of that monomial.



Comparing the two monomial lattices, we see that any monomial that is conventionally divisible by the Gröbner Basis is uniquely involutively divisible by the Involutive Basis. In other words, the involutive cones of the Involutive Basis form a disjoint cover of the conventional cones of the Gröbner Basis.

Fast Reduction

In the commutative case, we can sometimes use the properties of an involutive division to speed up the process of involutively reducing a polynomial with respect to a set of polynomials. For example, the Janet tree [27, 28] enables us to quickly determine whether a polynomial is involutively reducible by a set of polynomials with respect to the Janet involutive division.

In the noncommutative case, we usually use Algorithm 10 to involutively reduce a polynomial p with respect to a set of polynomials P. When this is done with respect to the left or right divisions however, we can improve Algorithm 10 by taking advantage of the fact that a monomial u_1 only involutively divides another monomial u_2 with respect to the left (right) division if u_1 is a suffix (prefix) of u_2.

For the left division, we can replace the code found in the first **if** loop of Algorithm 10 with the following code in order to obtain an improved algorithm.

```
if (LM(p_j) is a suffix of u) then
    found = true;
    p = p − (cLC(p_j)^{−1})u_ℓ p_j, where u_ℓ = Prefix(p, deg(p) − deg(p_j));
else
    j = j + 1;
end if
```

We note that only one operation is required to determine whether the monomial $\mathrm{LM}(p_j)$ involutively divides the monomial u here (test to see if $\mathrm{LM}(p_j)$ is a suffix of u); whereas in general there are many ways that $\mathrm{LM}(p_j)$ can conventionally divide u, each of which has to be tested to see whether it is an involutive reduction. This means that, with respect to the left or right divisions, we can determine whether a monomial u is involutively irreducible with respect to a set of polynomials P in linear time (linear in the number of elements in P); whereas in general we can only do this in quadratic time.

5.5.2 An Overlap-Based Local Division

Even though the left and right involutive divisions are strong and continuous (so that any Locally Involutive Basis returned by Algorithm 12 is a noncommutative Gröbner Basis), these divisions are not conclusive as the following example demonstrates.

Example 5.5.9 Let $F := \{xy - z,\ x + z,\ yz - z,\ xz,\ zy + z,\ z^2\}$ be a basis over the polynomial ring $\mathbb{Q}\langle x, y, z\rangle$, and let the monomial ordering be DegLex. Applying Algorithm 5 to F, we discover that F is a noncommutative Gröbner Basis (F is returned to us as the output of Algorithm 5). When we apply Algorithm 12 to F with respect to the left involutive division however, we notice that the algorithm goes into an infinite loop, constructing the infinite basis $G := F \cup \{zy^n - z,\ xy^n + z, zy^m + z, xy^m - z\}$, where $n \geqslant 2$, n even and $m \geqslant 3$, m odd.

The reason why Algorithm 12 goes into an infinite loop in the above example is that the right prolongations of the polynomials $xy - z$ and $zy + z$ by the variable y do not involutively reduce to zero (they reduce to the polynomials $xy^2 + z$ and $zy^2 - z$ respectively). These prolongations are the only prolongations of elements of F that do not involutively reduce to zero, and this is also true for all polynomials we subsequently add to F, thus allowing Algorithm 12 to construct the infinite set G.

Consider a modification of the left division where we assign the variable y to be right multiplicative for the (lead) monomials xy and zy. Then it is clear that F will be a Locally Involutive Basis with respect to this modified division, but will it also be true that F is an Involutive Basis and (had we not known so already) a Gröbner Basis?

Intuitively, for this particular example, it would seem that the answer to both of the above questions should be affirmative, because the modified division still ensures that all the overlap words associated with the S-polynomials of F are involutively irreducible (as placed in the overlap word) by at least one of the polynomials associated to each S-polynomial. This leads to the following idea for a *local* involutive division, where we refine the left division by choosing right nonmultiplicative variables based on the overlap words of S-polynomials associated to a set of polynomials only (note that there will also be a similar local involutive division refining the right division called the right overlap division).

Definition 5.5.10 (The Left Overlap Division $\mathcal{O}$) Let $U = \{u_1, \ldots, u_m\}$ be a set of monomials, and assume that all variables are left and right multiplicative for all elements of U to begin with.

(a) For all possible ways that a monomial $u_j \in U$ is a subword of a (different) monomial $u_i \in U$, so that

$$\mathrm{Subword}(u_i, k, k + \deg(u_j) - 1) = u_j$$

for some integer k, if u_j is not a suffix of u_i, assign the variable $\mathrm{Subword}(u_i, k + \deg(u_j), k + \deg(u_j))$ to be right nonmultiplicative for u_j.

(b) For all possible ways that a proper prefix of a monomial $u_i \in U$ is equal to a proper suffix of a (not necessarily different) monomial $u_j \in U$, so that

$$\mathrm{Prefix}(u_i, k) = \mathrm{Suffix}(u_j, k)$$

for some integer k and u_i is not a subword of u_j or vice-versa, assign the variable $\text{Subword}(u_i, k+1, k+1)$ to be right nonmultiplicative for u_j.

Remark 5.5.11 One possible algorithm for the left overlap division is presented in Algorithm 13, where the reason for insisting that the input set of monomials is ordered with respect to DegRevLex is in order to minimise the number of operations needed to discover all the subword overlaps (a monomial of degree d_1 can never be a subword of a different monomial of degree $d_2 \leqslant d_1$).

Example 5.5.12 Consider again the set of polynomials $F := \{xy - z,\ x + z,\ yz - z,\ xz,\ zy + z,\ z^2\}$ from Example 5.5.9. Here are the left and right multiplicative variables for $\text{LM}(F)$ with respect to the left overlap division $\mathcal{O}$.

u	$\mathcal{M}^L_{\mathcal{O}}(u, \text{LM}(F))$	$\mathcal{M}^R_{\mathcal{O}}(u, \text{LM}(F))$
xy	$\{x, y, z\}$	$\{x, y\}$
x	$\{x, y, z\}$	$\{x\}$
yz	$\{x, y, z\}$	$\{x\}$
xz	$\{x, y, z\}$	$\{x\}$
zy	$\{x, y, z\}$	$\{x, y\}$
z^2	$\{x, y, z\}$	$\{x\}$

When we apply Algorithm 12 to F with respect to the DegLex monomial ordering and the left overlap division, F is returned to us as the output, an assertion that is easily verified by showing that the 10 right prolongations of elements of F all involutively reduce to zero using F. This means that F is a Locally Involutive Basis with respect to the left overlap division; to show that F (and indeed any Locally Involutive Basis returned by Algorithm 12 with respect to the left overlap division) is also an Involutive Basis with respect to the left overlap division, we need to show that the left overlap division is continuous and either strong or Gröbner; we begin (after the following remark) by showing that the left overlap division is continuous.

Remark 5.5.13 In the above example, the table of multiplicative variables can be constructed from the table T shown below, a table that is obtained by applying Algorithm 13 to $\text{LM}(F)$.

Algorithm 13 The Left Overlap Division $\mathcal{O}$

Input: A set of monomials $U = \{u_1, \ldots, u_m\}$ ordered by DegRevLex ($u_1 \geqslant u_2 \geqslant \cdots \geqslant u_m$), where $u_i \in R\langle x_1, \ldots, x_n\rangle$.

Output: A table T of left and right multiplicative variables for all $u_i \in U$, where each entry of T is either 1 (multiplicative) or 0 (nonmultiplicative).

Create a table T of multiplicative variables as shown below:

	x_1^L	x_1^R	x_2^L	x_2^R	$\cdots$	x_n^L	x_n^R
u_1	1	1	1	1	$\cdots$	1	1
u_2	1	1	1	1	$\cdots$	1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
u_m	1	1	1	1	$\cdots$	1	1

for each monomial $u_i \in U$ $(1 \leqslant i \leqslant m)$ **do**
  for each monomial $u_j \in U$ $(i \leqslant j \leqslant m)$ **do**
    Let $u_i = x_{i_1}x_{i_2}\ldots x_{i_\alpha}$ and $u_j = x_{j_1}x_{j_2}\ldots x_{j_\beta}$;
    if $(i \neq j)$ **then**
      for each k $(1 \leqslant k < \alpha - \beta + 1)$ **do**
        if (Subword$(u_i, k, k+\beta-1) == u_j$) **then**
          $T(u_j, x^R_{i_{k+\beta}}) = 0$;
        end if
      end for
    end if
    for each k $(1 \leqslant k \leqslant \beta - 1)$ **do**
      if (Prefix$(u_i, k) ==$ Suffix(u_j, k)) **then**
        $T(u_j, x^R_{i_{k+1}}) = 0$;
      end if
      if (Suffix$(u_i, k) ==$ Prefix(u_j, k)) **then**
        $T(u_i, x^R_{j_{k+1}}) = 0$;
      end if
    end for
  end for
end for
return T;

Monomial	x^L	x^R	y^L	y^R	z^L	z^R
xy	1	1	1	1	1	0
x	1	1	1	0	1	0
yz	1	1	1	0	1	0
xz	1	1	1	0	1	0
zy	1	1	1	1	1	0
z^2	1	1	1	0	1	0

The zero entries in T correspond to the following overlaps between the elements of $\mathrm{LM}(F)$.

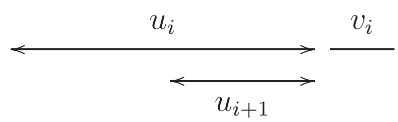
Table Entry	Overlap
$T(xy, z^R)$	$\mathrm{Suffix}(xy, 1) = \mathrm{Prefix}(yz, 1)$
$T(x, y^R)$	$\mathrm{Subword}(xy, 1, 1) = x$
$T(x, z^R)$	$\mathrm{Subword}(xz, 1, 1) = x$
$T(yz, y^R)$	$\mathrm{Suffix}(yz, 1) = \mathrm{Prefix}(zy, 1)$
$T(yz, z^R)$	$\mathrm{Suffix}(yz, 1) = \mathrm{Prefix}(z^2, 1)$
$T(xz, y^R)$	$\mathrm{Suffix}(xz, 1) = \mathrm{Prefix}(zy, 1)$
$T(xz, z^R)$	$\mathrm{Suffix}(xz, 1) = \mathrm{Prefix}(z^2, 1)$
$T(zy, z^R)$	$\mathrm{Suffix}(zy, 1) = \mathrm{Prefix}(yz, 1)$
$T(z^2, y^R)$	$\mathrm{Suffix}(z^2, 1) = \mathrm{Prefix}(zy, 1)$
$T(z^2, z^R)$	$\mathrm{Suffix}(z^2, 1) = \mathrm{Prefix}(z^2, 1)$

Proposition 5.5.14 *The left overlap division $\mathcal{O}$ is continuous.*

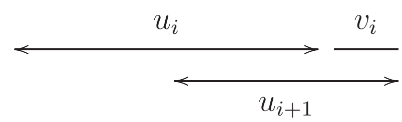
Proof: Let w be an arbitrary fixed monomial; let U be any set of monomials; and consider any sequence $(u_1,\ u_2,\ \ldots,\ u_k)$ of monomials from U ($u_i \in U$ for all $1 \leqslant i \leqslant k$), each of which is a conventional divisor of w (so that $w = \ell_i u_i r_i$ for all $1 \leqslant i \leqslant k$, where the ℓ_i and the r_i are monomials). For all $1 \leqslant i < k$, suppose that the monomial u_{i+1} satisfies condition (b) of Definition 5.4.2 (condition (a) can never be satisfied because $\mathcal{O}$ never assigns any left nonmultiplicative variables). To show that $\mathcal{O}$ is continuous, we must show that no two pairs (ℓ_i, r_i) and (ℓ_j, r_j) are the same, where $i \neq j$.

Consider an arbitrary monomial u_i from the sequence, where $1 \leqslant i < k$. By definition of $\mathcal{O}$, the next monomial u_{i+1} in the sequence cannot be either a prefix or a proper subword of u_i. This leaves two possibilities: (i) u_{i+1} is a suffix of u_i (in which case $\deg(u_{i+1}) < \deg(u_i)$); or (ii) u_{i+1} is a suffix of the prolongation $u_i v_i$ of u_i, where $v_i := \mathrm{Prefix}(r_i, 1)$.

Example of possibility (i)



Example of possibility (ii)



In both cases, it is clear that we have $\deg(r_{i+1}) \leqslant \deg(r_i)$, so that $\deg(r_1) \geqslant \deg(r_2) \geqslant \cdots \geqslant \deg(r_k)$. It follows that no two (ℓ, r) pairs in the sequence can be the same, because for each subsequence $u_a, u_{a+1}, \ldots, u_b$ such that $\deg(r_a) = \deg(r_{a+1}) = \cdots = \deg(r_b)$, we must have $\deg(\ell_a) < \deg(\ell_{a+1}) < \cdots < \deg(\ell_b)$. □

Having shown that the left overlap division is continuous, one way of showing that every Locally Involutive Basis with respect to the left overlap division is an Involutive Basis would be to show that the left overlap division is a strong involutive division. However, the left overlap division is only a weak involutive division, as the following counterexample demonstrates.

Proposition 5.5.15 *The left overlap division is a weak involutive division.*

Proof: Let $U := \{yz, xy\}$ be a set of monomials over the polynomial ring $\mathbb{Q}\langle x, y, z\rangle$. Here are the multiplicative variables for U with respect to the left overlap division $\mathcal{O}$.

u	$\mathcal{M}^L_{\mathcal{O}}(u, U)$	$\mathcal{M}^R_{\mathcal{O}}(u, U)$
yz	$\{x, y, z\}$	$\{x, y, z\}$
xy	$\{x, y, z\}$	$\{x, y\}$

Because $yzxy \in \mathcal{C}_{\mathcal{O}}(yz, U)$ and $yzxy \in \mathcal{C}_{\mathcal{O}}(xy, U)$, one of the conditions $\mathcal{C}_{\mathcal{O}}(yz, U) \subset \mathcal{C}_{\mathcal{O}}(xy, U)$ or $\mathcal{C}_{\mathcal{O}}(xy, U) \subset \mathcal{C}_{\mathcal{O}}(yz, U)$ must be satisfied in order for $\mathcal{O}$ to be a strong involutive division (this is the Disjoint Cones condition of Definition 5.1.6). But neither of these conditions can be satisfied when we consider that $xy \notin \mathcal{C}_{\mathcal{O}}(yz, U)$ and $yz \notin \mathcal{C}_{\mathcal{O}}(xy, U)$, so $\mathcal{O}$ must be a weak involutive division. □

The weakness of the left overlap division is the price we pay for refining the left division by allowing more right multiplicative variables. All is not lost however, as we can still show that every Locally Involutive Basis with respect to the left overlap division is an Involutive Basis by showing that the left overlap division is a Gröbner involutive division.

Proposition 5.5.16 *The left overlap division $\mathcal{O}$ is a Gröbner involutive division.*

Proof: We are required to show that if Algorithm 12 terminates with $\mathcal{O}$ and some arbitrary admissible monomial ordering O as input, then the Locally Involutive Basis G it returns is a noncommutative Gröbner Basis. By Definition 3.1.8, we can do this by showing that all S-polynomials involving elements of G conventionally reduce to zero using G.

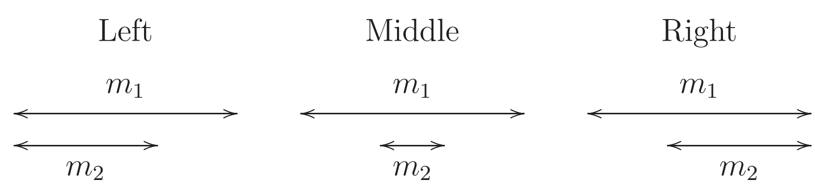
Assume that $G = \{g_1, \ldots, g_p\}$ is sorted (by lead monomial) with respect to the DegRevLex monomial ordering (greatest first), and let $U = \{u_1, \ldots, u_p\} := \{\mathrm{LM}(g_1), \ldots, \mathrm{LM}(g_p)\}$ be the set of leading monomials. Let T be the table obtained by applying Algorithm 13 to U. Because G is a Locally Involutive Basis, every zero entry $T(u_i, x_j^\Gamma)$ $(\Gamma \in \{L, R\})$ in the table corresponds to a prolongation $g_i x_j$ or $x_j g_i$ that involutively reduces to zero.

Let S be the set of S-polynomials involving elements of G, where the t-th entry of S $(1 \leqslant t \leqslant |S|)$ is the S-polynomial

$$s_t = c_t \ell_t g_i r_t - c'_t \ell'_t g_j r'_t,$$

with $\ell_t u_i r_t = \ell'_t u_j r'_t$ being the overlap word of the S-polynomial. We will prove that every S-polynomial in S conventionally reduces to zero using G.

Recall (from Definition 3.1.2) that each S-polynomial in S corresponds to a particular type of overlap — 'prefix', 'subword' or 'suffix'. For the purposes of this proof, let us now split the subword overlaps into three further types — 'left', 'middle' and 'right', corresponding to the cases where a monomial m_2 is a prefix, proper subword and suffix of a monomial m_1.



This classification provides us with five cases to deal with in total, which we shall process in the following order: right, middle, left, prefix, suffix.

(1) Consider an arbitrary entry $s_t \in S$ $(1 \leqslant t \leqslant |S|)$ corresponding to a right overlap where the monomial u_j is a suffix of the monomial u_i. Because $\mathcal{O}$ never assigns any left nonmultiplicative variables, u_j must be an involutive divisor of u_i. But this contradicts the fact that the set G is autoreduced; it follows that no S-polynomials corresponding to

right overlaps can appear in S.

(2) Consider an arbitrary entry $s_t \in S$ $(1 \leqslant t \leqslant |S|)$ corresponding to a middle overlap where the monomial u_j is a proper subword of the monomial u_i. This means that $s_t = c_t g_i - c'_t \ell'_t g_j r'_t$ for some $g_i, g_j \in G$, with overlap word $u_i = \ell'_t u_j r'_t$. Let $u_i = x_{i_1} \ldots x_{i_\alpha}$; let $u_j = x_{j_1} \ldots x_{j_\beta}$; and choose D such that $x_{i_D} = x_{j_\beta}$.

$$
\begin{array}{lccccccccccc}
u_i = & x_{i_1} & - - - & x_{i_{D-\beta}} & x_{i_{D-\beta+1}} & x_{i_{D-\beta+2}} & - - - & x_{i_{D-1}} & x_{i_D} & x_{i_{D+1}} & - - - & x_{i_\alpha} \\
u_j = & & & & x_{j_1} & x_{j_2} & - - - & x_{j_{\beta-1}} & x_{j_\beta} & & &
\end{array}
$$

Because u_j is a proper subword of u_i, it follows that $T(u_j, x^R_{i_{D+1}}) = 0$. This gives rise to the prolongation $g_j x_{i_{D+1}}$ of g_j. But we know that all prolongations involutively reduce to zero (G is a Locally Involutive Basis), so Algorithm 10 must find a monomial $u_k = x_{k_1} \ldots x_{k_\gamma} \in U$ such that u_k involutively divides $u_j x_{i_{D+1}}$. Assuming that $x_{k_\gamma} = x_{i_\kappa}$, we can deduce that any candidate for u_k must be a suffix of $u_j x_{i_{D+1}}$ (otherwise $T(u_k, x^R_{i_{\kappa+1}}) = 0$ because of the overlap between u_i and u_k). This means that the degree of u_k is in the range $1 \leqslant \gamma \leqslant \beta + 1$; we shall illustrate this in the following diagram by using a squiggly line to indicate that the monomial u_k can begin anywhere (or nowhere if $u_k = x_{i_{D+1}}$) on the squiggly line.

$$
\begin{array}{lccccccccccc}
u_i = & x_{i_1} & - - - & x_{i_{D-\beta}} & x_{i_{D-\beta+1}} & x_{i_{D-\beta+2}} & - - - & x_{i_{D-1}} & x_{i_D} & x_{i_{D+1}} & - - - & x_{i_\alpha} \\
u_j = & & & & x_{j_1} & x_{j_2} & - - - & x_{j_{\beta-1}} & x_{j_\beta} & & & \\
u_k = & & & & \sim\sim\sim & \sim\sim\sim & \sim\sim\sim & \sim\sim\sim & \sim\sim\sim & x_{k_\gamma} & &
\end{array}
$$

We can now use the monomial u_k together with Buchberger's Second Criterion to simplify our goal of showing that the S-polynomial s_t reduces to zero. Notice that the monomial u_k is a subword of the overlap word u_i associated to s_t, and so in order to show that s_t reduces to zero, all we have to do is to show that the two S-polynomials

$$s_u = c_u g_i - c'_u (x_{i_1} x_{i_2} \ldots x_{i_{D+1-\gamma}}) g_k (x_{i_{D+2}} \ldots x_{i_\alpha})$$

and[2]

$$s_v = c_v (x_{j_1} \ldots x_{i_{D+1-\gamma}}) g_k - c'_v g_j x_{i_{D+1}}$$

[2]Technical point: if $\gamma \neq \beta + 1$, the S-polynomial s_v could in fact appear as $s_v = c_v g_j x_{i_{D+1}} - c'_v (x_{j_1} \ldots x_{i_{D+1-\gamma}}) g_k$ and not as $s_v = c_v (x_{j_1} \ldots x_{i_{D+1-\gamma}}) g_k - c'_v g_j x_{i_{D+1}}$; for simplicity we will treat both cases the same in the proof as all that changes is the notation and the signs.

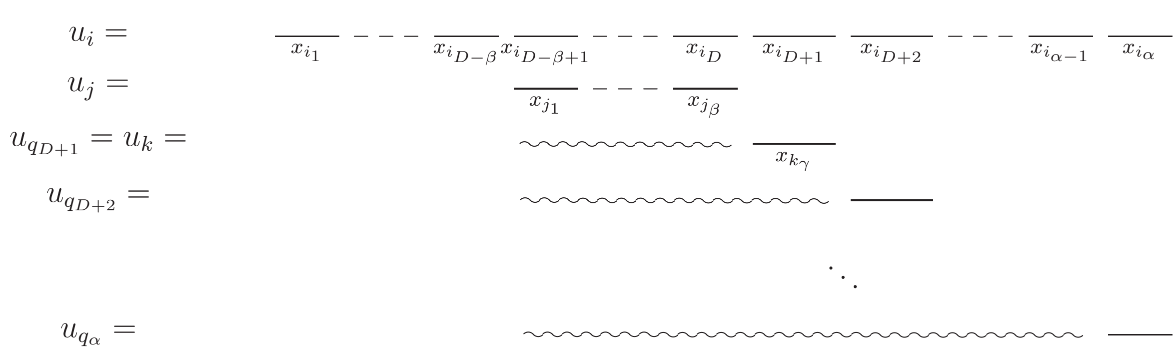
reduce to zero ($1 \leqslant u, v \leqslant |S|$).

For the S-polynomial s_v, there are two cases to consider: $\gamma = 1$, and $\gamma > 1$. In the former case, because (as placed in u_i) the monomials u_j and u_k do not overlap, we can use Buchberger's First Criterion to say that the 'S-polynomial' s_v reduces to zero (for further explanation, see the paragraph at the beginning of Section 3.4.1). In the latter case, we know that the first step of the involutive reduction of the prolongation $g_j x_{i_{D+1}}$ is to take away the multiple $(\frac{c_v}{c'_v})(x_{j_1} \dots x_{i_{D+1-\gamma}})g_k$ of g_k from $g_j x_{i_{D+1}}$ to leave the polynomial $g_j x_{i_{D+1}} - (\frac{c_v}{c'_v})(x_{j_1} \dots x_{i_{D+1-\gamma}})g_k = -(\frac{1}{c'_v})s_v$. But as we know that all prolongations involutively reduce to zero, we can conclude that the S-polynomial s_v conventionally reduces to zero.

For the S-polynomial s_u, we note that if $D = \alpha - 1$, then s_u corresponds to a right overlap. But we know from part (1) that right overlaps cannot appear in S, and so s_t also cannot appear in S. Otherwise, we proceed by induction on the S-polynomial s_u to produce a sequence $\{u_{q_{D+1}}, u_{q_{D+2}}, \dots, u_{q_\alpha}\}$ of monomials, so that s_u (and hence s_t) reduces to zero if the S-polynomial

$$s_\eta = c_\eta g_i - c'_\eta (x_{i_1} \dots x_{i_{\alpha-\mu}}) g_{q_\alpha}$$

reduces to zero ($1 \leqslant \eta \leqslant |S|$), where $\mu = \deg(u_{q_\alpha})$.



But s_η always corresponds to a right overlap, so we must conclude that middle overlaps (as well as right overlaps) cannot appear in S.

(3) Consider an arbitrary entry $s_t \in S$ ($1 \leqslant t \leqslant |S|$) corresponding to a left overlap where the monomial u_j is a prefix of the monomial u_i. This means that $s_t = c_t g_i - c'_t g_j r'_t$ for

some $g_i, g_j \in G$, with overlap word $u_i = u_j r'_t$. Let $u_i = x_{i_1} \ldots x_{i_\alpha}$ and let $u_j = x_{j_1} \ldots x_{j_\beta}$.

$$
\begin{array}{lccccccccc}
u_i = & \overline{x_{i_1}} & \overline{x_{i_2}} & --- & \overline{x_{i_{\beta-1}}} & \overline{x_{i_\beta}} & \overline{x_{i_{\beta+1}}} & --- & \overline{x_{i_{\alpha-1}}} & \overline{x_{i_\alpha}} \\
u_j = & \overline{x_{j_1}} & \overline{x_{j_2}} & --- & \overline{x_{j_{\beta-1}}} & \overline{x_{j_\beta}} & & & &
\end{array}
$$

Because u_j is a prefix of u_i, it follows that $T(u_j, x^R_{i_{\beta+1}}) = 0$. This gives rise to the prolongation $g_j x_{i_{\beta+1}}$ of g_j. But we know that all prolongations involutively reduce to zero, so there must exist a monomial $u_k = x_{k_1} \ldots x_{k_\gamma} \in U$ such that u_k involutively divides $u_j x_{i_{\beta+1}}$. Assuming that $x_{k_\gamma} = x_{i_\kappa}$, any candidate for u_k must be a suffix of $u_j x_{i_{\beta+1}}$ (otherwise $T(u_k, x^R_{i_{\kappa+1}}) = 0$ because of the overlap between u_i and u_k). Further, any candidate for u_k cannot be either a suffix or a proper subword of u_i (because of parts (1) and (2) of this proof). This leaves only one possibility for u_k, namely $u_k = u_j x_{i_{\beta+1}}$.

$$
\begin{array}{lccccccccc}
u_i = & \overline{x_{i_1}} & \overline{x_{i_2}} & --- & \overline{x_{i_{\beta-1}}} & \overline{x_{i_\beta}} & \overline{x_{i_{\beta+1}}} & --- & \overline{x_{i_{\alpha-1}}} & \overline{x_{i_\alpha}} \\
u_j = & \overline{x_{j_1}} & \overline{x_{j_2}} & --- & \overline{x_{j_{\beta-1}}} & \overline{x_{j_\beta}} & & & & \\
u_k = & \overline{x_{k_1}} & \overline{x_{k_2}} & --- & \overline{x_{k_{\gamma-2}}} & \overline{x_{k_{\gamma-1}}} & \overline{x_{k_\gamma}} & & &
\end{array}
$$

If $\alpha = \beta + 1$, then it is clear that $u_k = u_i$, and so the first step in the involutive reduction of the prolongation $g_j x_{i_\alpha}$ is to take away the multiple $(\frac{c_t}{c'_t}) g_i$ of g_i from $g_j x_{i_\alpha}$ to leave the polynomial $g_j x_{i_\alpha} - (\frac{c_t}{c'_t}) g_i = -(\frac{1}{c'_t}) s_t$. But as we know that all prolongations involutively reduce to zero, we can conclude that the S-polynomial s_t conventionally reduces to zero.

Otherwise, if $\alpha > \beta + 1$, we can now use the monomial u_k together with Buchberger's Second Criterion to simplify our goal of showing that the S-polynomial s_t reduces to zero. Notice that the monomial u_k is a subword of the overlap word u_i associated to s_t, and so in order to show that s_t reduces to zero, all we have to do is to show that the two S-polynomials

$$s_u = c_u g_i - c'_u g_k (x_{i_{\beta+2}} \ldots x_{i_\alpha})$$

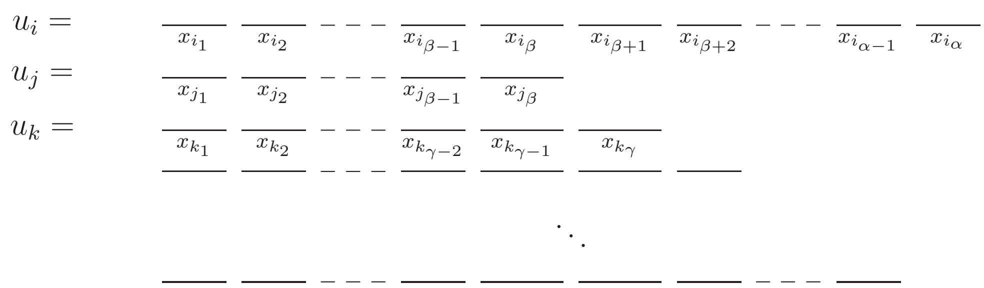
and

$$s_v = c_v g_k - c'_v g_j x_{i_{\beta+1}}$$

reduce to zero ($1 \leqslant u, v \leqslant |S|$).

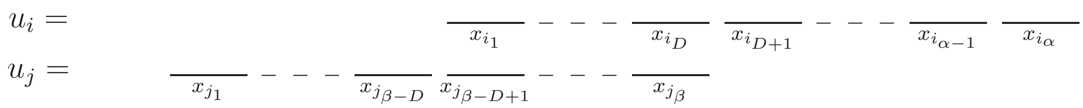
The S-polynomial s_v reduces to zero by comparison with part (2). For the S-polynomial s_u, we proceed by induction (we have another left overlap), eventually coming across a left overlap of 'type $\alpha = \beta + 1$' because we move one letter at a time to the right after

each inductive step.



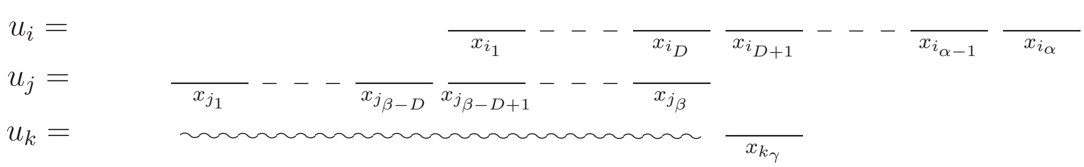
(4 and 5) In Definition 3.1.2, we defined a prefix overlap to be an overlap where, given two monomials m_1 and m_2 such that $\deg(m_1) \geqslant \deg(m_2)$, a prefix of m_1 is equal to a suffix of m_2; suffix overlaps were defined similarly. If we drop the condition on the degrees of the monomials, it is clear that every suffix overlap can be treated as a prefix overlap (by swapping the roles of m_1 and m_2); this allows us to deal with the case of a prefix overlap only.

Consider an arbitrary entry $s_t \in S$ $(1 \leqslant t \leqslant |S|)$ corresponding to a prefix overlap where a prefix of the monomial u_i is equal to a suffix of the monomial u_j. This means that $s_t = c_t \ell_t g_i - c'_t g_j r'_t$ for some $g_i, g_j \in G$, with overlap word $\ell_t u_i = u_j r'_t$. Let $u_i = x_{i_1} \dots x_{i_\alpha}$; let $u_j = x_{j_1} \dots x_{j_\beta}$; and choose D such that $x_{i_D} = x_{j_\beta}$.



By definition of $\mathcal{O}$, we must have $T(u_j, x^R_{i_{D+1}}) = 0$.

Because we know that the prolongation $g_j x_{i_{D+1}}$ involutively reduces to zero, there must exist a monomial $u_k = x_{k_1} \dots x_{k_\gamma} \in U$ such that u_k involutively divides $u_j x_{i_{D+1}}$. This u_k must be a suffix of $u_j x_{i_{D+1}}$ (otherwise, assuming that $x_{k_\gamma} = x_{j_\kappa}$, we have $T(u_k, x^R_{i_{D+1}}) = 0$ if $\kappa = \beta$ (because of the overlap between u_i and u_k); and $T(u_k, x^R_{j_{\kappa+1}}) = 0$ if $\kappa < \beta$ (because of the overlap between u_j and u_k)).



Let us now use the monomial u_k together with Buchberger's Second Criterion to simplify our goal of showing that the S-polynomial s_t reduces to zero. Because u_k is a subword of the overlap word $\ell_t u_i$ associated to s_t, in order to show that s_t reduces to zero, all we have to do is to show that the two S-polynomials

$$s_u = \begin{cases} c_u(x_{k_1} \dots x_{j_{\beta-D}})g_i - c'_u g_k(x_{i_{D+2}} \dots x_{i_\alpha}) & \text{if } \gamma > D+1 \\ c_u g_i - c'_u \ell'_u g_k(x_{i_{D+2}} \dots x_{i_\alpha}) & \text{if } \gamma \leqslant D+1 \end{cases}$$

and

$$s_v = c_v g_j x_{i_{D+1}} - c'_v(x_{j_1} \dots x_{j_{\beta+1-\gamma}})g_k$$

reduce to zero ($1 \leqslant u, v \leqslant |S|$).

The S-polynomial s_v reduces to zero by comparison with part (2). For the S-polynomial s_u, first note that if $\alpha = D+1$, then either u_k is a suffix of u_i, u_i is a suffix of u_k, or $u_k = u_i$; it follows that s_u reduces to zero trivially if $u_k = u_i$, and (by part (1)) s_u (and hence s_t) cannot appear in S in the other two cases.

If however $\alpha \neq D+1$, then either s_u is a middle overlap (if $\gamma < D+1$), a left overlap (if $\gamma = D+1$), or another prefix overlap. The first case leads us to conclude that s_t cannot appear in S; the second case is handled by part (3) of this proof; and the final case is handled by induction, where we note that after each step of the induction, the value $\alpha + \beta - 2D$ strictly decreases, so we are guaranteed at some stage to find an overlap that is not a prefix overlap, enabling us either to verify that the S-polynomial s_t conventionally reduces to zero, or to conclude that s_t can not in fact appear in S. □

5.5.3 A Strong Local Division

Thus far, we have encountered two global divisions that are strong and continuous, and one local division that is weak, continuous and Gröbner. Our next division can be considered to be a hybrid of these previous divisions, as it will be a local division that is continuous and (as long as thick divisors are being used) strong.

Definition 5.5.17 (The Strong Left Overlap Division $\mathcal{S}$) Let $U = \{u_1, \dots, u_m\}$ be a set of monomials. Assign multiplicative variables to U according to Algorithm 15, which (in words) performs the following two tasks.

(a) Assign multiplicative variables to U according to the left overlap division.

(b) Using the recipe provided in Algorithm 14, ensure that at least one variable in every monomial $u_j \in U$ is right nonmultiplicative for each monomial $u_i \in U$.

Remark 5.5.18 As Algorithm 15 expects any input set to be ordered with respect to DegRevLex, we may sometimes have to reorder a set of monomials U to satisfy this condition before we can assign multiplicative variables to U according to the strong left overlap division.

Algorithm 14 'DisjointCones' Function for Algorithm 15

Input: A set of monomials $U = \{u_1, \ldots, u_m\}$ ordered by DegRevLex ($u_1 \geqslant u_2 \geqslant \cdots \geqslant u_m$), where $u_i \in R\langle x_1, \ldots, x_n\rangle$; a table T of left and right multiplicative variables for all $u_i \in U$, where each entry of T is either 1 (multiplicative) or 0 (nonmultiplicative).

Output: T.

for each monomial $u_i \in U$ $(m \geqslant i \geqslant 1)$ **do**
  for each monomial $u_j \in U$ $(m \geqslant j \geqslant 1)$ **do**
    Let $u_i = x_{i_1}x_{i_2}\ldots x_{i_\alpha}$ and $u_j = x_{j_1}x_{j_2}\ldots x_{j_\beta}$;
    found = false;
    $k = 1$;
    while $(k \leqslant \beta)$ **do**
      if $(T(u_i, x^R_{j_k}) = 0)$ **then**
        found = true;
        $k = \beta + 1$;
      else
        $k = k + 1$;
      end if
    end while
    if (found == false) **then**
      $T(u_i, x^R_{j_1}) = 0$;
    end if
  end for
end for
return T;

Algorithm 15 The Strong Left Overlap Division $\mathcal{S}$

Input: A set of monomials $U = \{u_1, \ldots, u_m\}$ ordered by DegRevLex ($u_1 \geqslant u_2 \geqslant \cdots \geqslant u_m$), where $u_i \in R\langle x_1, \ldots, x_n\rangle$.

Output: A table T of left and right multiplicative variables for all $u_i \in U$, where each entry of T is either 1 (multiplicative) or 0 (nonmultiplicative).

Create a table T of multiplicative variables as shown below:

	x_1^L	x_1^R	x_2^L	x_2^R	$\cdots$	x_n^L	x_n^R
u_1	1	1	1	1	$\cdots$	1	1
u_2	1	1	1	1	$\cdots$	1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
u_m	1	1	1	1	$\cdots$	1	1

for each monomial $u_i \in U$ $(1 \leqslant i \leqslant m)$ **do**
  for each monomial $u_j \in U$ $(i \leqslant j \leqslant m)$ **do**
    Let $u_i = x_{i_1}x_{i_2}\ldots x_{i_\alpha}$ and $u_j = x_{j_1}x_{j_2}\ldots x_{j_\beta}$;
    if $(i \neq j)$ **then**
      for each k $(1 \leqslant k < \alpha - \beta + 1)$ **do**
        if (Subword$(u_i, k, k+\beta-1) == u_j$) **then**
          $T(u_j, x^R_{i_{k+\beta}}) = 0$;
        end if
      end for
    end if
    for each k $(1 \leqslant k \leqslant \beta - 1)$ **do**
      if (Prefix$(u_i, k) ==$ Suffix(u_j, k)) **then**
        $T(u_j, x^R_{i_{k+1}}) = 0$;
      end if
      if (Suffix$(u_i, k) ==$ Prefix(u_j, k)) **then**
        $T(u_i, x^R_{j_{k+1}}) = 0$;
      end if
    end for
  end for
end for
$T = \text{DisjointCones}(U, T)$; *(Algorithm 14)*
return T;

Proposition 5.5.19 *The strong left overlap division is continuous.*

Proof: We refer to the proof of Proposition 5.5.14, replacing $\mathcal{O}$ by $\mathcal{S}$. $\square$

Proposition 5.5.20 *The strong left overlap division is a Gröbner involutive division.*

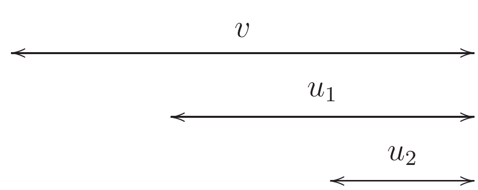
Proof: We refer to the proof of Proposition 5.5.16, replacing $\mathcal{O}$ by $\mathcal{S}$. $\square$

Remark 5.5.21 Propositions 5.5.19 and 5.5.20 apply either when using thin divisors or when using thick divisors.

Proposition 5.5.22 *With respect to thick divisors, the strong left overlap division is a strong involutive division.*

Proof: To prove that the strong left overlap division is a strong involutive division, we need to show that the three conditions of Definition 5.1.6 hold.

- **Disjoint Cones Condition**
  Let $\mathcal{C}_\mathcal{S}(u_1, U)$ and $\mathcal{C}_\mathcal{S}(u_2, U)$ be the involutive cones associated to the monomials u_1 and u_2 over some noncommutative polynomial ring $\mathcal{R}$, where $\{u_1, u_2\} \subset U \subset \mathcal{R}$. If $\mathcal{C}_\mathcal{S}(u_1, U) \cap \mathcal{C}_\mathcal{S}(u_2, U) \neq \emptyset$, then there must be some monomial $v \in \mathcal{R}$ such that v contains both monomials u_1 and u_2 as subwords, and (as placed in v) both u_1 and u_2 must be involutive divisors of v. By definition of $\mathcal{S}$, both u_1 and u_2 must be suffices of v. Thus, assuming (without loss of generality) that $\deg(u_1) > \deg(u_2)$, we are able to draw the following diagram summarising the situation.

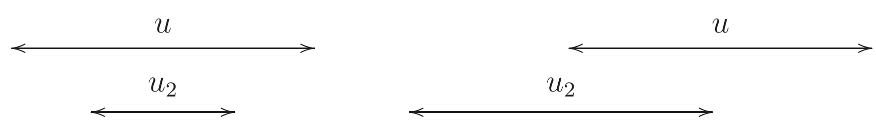


  For $\mathcal{S}$ to be strong, we must have $\mathcal{C}_\mathcal{S}(u_1, U) \subset \mathcal{C}_\mathcal{S}(u_2, U)$ (it is clear that $\mathcal{C}_\mathcal{S}(u_2, U) \not\subset \mathcal{C}_\mathcal{S}(u_1, U)$ because $u_2 \notin \mathcal{C}_\mathcal{S}(u_1, U)$). This can be verified by proving that a variable is right nonmultiplicative for u_1 if and only if it is right nonmultiplicative for u_2.

  ($\Rightarrow$) If an arbitrary variable x is right nonmultiplicative for u_2, then either some monomial $u \in U$ overlaps with u_2 in one of the ways shown below (where the variable immediately to the right of u_2 is the variable x), or x was assigned right

nonmultiplicative for u_2 in order to ensure that some variable in some monomial $u \in U$ is right nonmultiplicative for u_2.

Overlap (i) Overlap (ii)



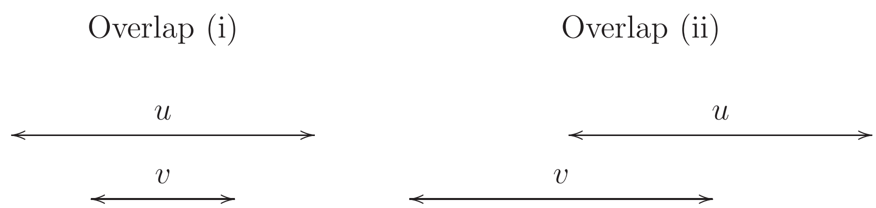
If the former case applies, then it is clear that for both overlap types there will be another overlap between u_1 and u that will lead $\mathcal{S}$ to assign x to be right nonmultiplicative for u_1. It follows that after we have assigned multiplicative variables to U according to the left overlap division (which we recall is the first step of assigning multiplicative variables to U according to $\mathcal{S}$), the right multiplicative variables of u_1 and u_2 will be identical. It therefore remains to show that if x is assigned right nonmultiplicative for u_2 in the latter case (which will happen during the final step of assigning multiplicative variables to U according to $\mathcal{S}$), then x is also assigned right nonmultiplicative for u_1. But this is clear when we consider that Algorithm 14 is used to perform this final step, because for u_1 and u_2 in Algorithm 14, we will always analyse each monomial in U in the same order.

($\Leftarrow$) Use the same argument as above, replacing u_1 by u_2 and vice-versa.

- **Unique Divisor Condition**
  Given a monomial u belonging to a set of monomials U, u may not involutively divide an arbitrary monomial v in more than one way (and hence the Unique Divisor condition is satisfied) because (i) $\mathcal{S}$ ensures that no overlap word involving only u is involutively divisible in more than one way by u; and (ii) $\mathcal{S}$ ensures that at least one variable in u is right nonmultiplicative for u, so that if u appears twice in v as subwords that are disjoint from one another, then only the 'right-most' subword can potentially be an involutive divisor of v.

- **Subset Condition**
  Let v be a monomial belonging to a set V of monomials, where V itself is a subset of a larger set U of monomials. Because $\mathcal{S}$ assigns no left nonmultiplicative variables, it is clear that $\mathcal{M}^L_{\mathcal{S}}(v, U) \subseteq \mathcal{M}^L_{\mathcal{S}}(v, V)$. To prove that $\mathcal{M}^R_{\mathcal{S}}(v, U) \subseteq \mathcal{M}^R_{\mathcal{S}}(v, V)$, note that if a variable x is right nonmultiplicative for v with respect to U and $\mathcal{S}$ (so that $x \notin \mathcal{M}^R_{\mathcal{S}}(v, U)$), then (as in the proof for the Disjoint Cones Condition) either some monomial $u \in U$ overlaps with v in one of the ways shown below (where the

variable immediately to the right of v is the variable x), or x was assigned right nonmultiplicative for v in order to ensure that some variable in some monomial $u \in U$ is right nonmultiplicative for v.



In both cases, it is clear that, with respect to the set V, the variable x may not be assigned right nonmultiplicative for v if $u \notin V$, so that $\mathcal{M}^R_{\mathcal{S}}(v, U) \subseteq \mathcal{M}^R_{\mathcal{S}}(v, V)$ as required.

$\square$

Proposition 5.5.23 *With respect to thin divisors, the strong left overlap division is a weak involutive division.*

Proof: Let $U := \{xy\}$ be a set of monomials over the polynomial ring $\mathbb{Q}\langle x, y\rangle$. Here are the multiplicative variables for U with respect to the strong left overlap division $\mathcal{S}$.

u	$\mathcal{M}^L_{\mathcal{S}}(u, U)$	$\mathcal{M}^R_{\mathcal{S}}(u, U)$
xy	$\{x, y\}$	$\{y\}$

For $\mathcal{S}$ to be strong with respect to thin divisors, the monomial xy^2xy, which is conventionally divisible by xy in two ways, must only be involutively divisible by xy in one way (this is the Unique Divisor condition of Definition 5.1.6). However it is clear that xy^2xy is involutively divisible by xy in two ways with respect to thin divisors, so $\mathcal{S}$ must be a weak involutive division with respect to thin divisors. $\square$

Example 5.5.24 Continuing Examples 5.5.9 and 5.5.12, here are the multiplicative variables for the set $\mathrm{LM}(F)$ of monomials with respect to the strong left overlap division $\mathcal{S}$, where we recall that $F := \{xy - z,\ x + z,\ yz - z,\ xz,\ zy + z,\ z^2\}$.

u	$\mathcal{M}^L_{\mathcal{S}}(u, \mathrm{LM}(F))$	$\mathcal{M}^R_{\mathcal{S}}(u, \mathrm{LM}(F))$
xy	$\{x, y, z\}$	$\{y\}$
x	$\{x, y, z\}$	$\emptyset$
yz	$\{x, y, z\}$	$\emptyset$
xz	$\{x, y, z\}$	$\emptyset$
zy	$\{x, y, z\}$	$\{y\}$
z^2	$\{x, y, z\}$	$\emptyset$

When we apply Algorithm 12 to F with respect to the DegLex monomial ordering, thick divisors and the strong left overlap division, F (as in Example 5.5.12) is returned to us as the output Locally Involutive Basis.

Remark 5.5.25 In the above example, even though we know that $\mathcal{S}$ is continuous, we cannot deduce that the Locally Involutive Basis F is an Involutive Basis because we are using thick divisors (Proposition 5.4.3 does not apply in the case of using thick divisors).

What this means is that the involutive cones of F (and in general any Locally Involutive Basis with respect to $\mathcal{S}$ and thick divisors) will be disjoint (because $\mathcal{S}$ is strong), but will not necessarily completely cover the conventional cones of F, so that some monomials that are conventionally reducible by F may not be involutively reducible by F. It follows that when involutively reducing a polynomial with respect to F, the reduction path will be unique but the correct remainder may not always be obtained (in the sense that some of the terms in our 'remainder' may still be conventionally reducible by members of F). One remedy to this problem would be to involutively reduce a polynomial p with respect to F to obtain a remainder r, and then to conventionally reduce r with respect to F to obtain a remainder r' which we can be sure contains no term that is conventionally reducible by F.

Let us now summarise (with respect to thin divisors) the properties of the involutive divisions we have encountered so far, where we note that any strong and continuous involutive division is by default a Gröbner involutive division.

Division	Continuous	Strong	Gröbner
Left	Yes	Yes	Yes
Right	Yes	Yes	Yes
Left Overlap	Yes	No	Yes
Right Overlap	Yes	No	Yes
Strong Left Overlap	Yes	No	Yes
Strong Right Overlap	Yes	No	Yes

There is a balance to be struck between choosing an involutive division with nice theoretical properties and an involutive division which is of practical use, which is to say that it is more likely to terminate compared to other divisions. To this end, one suggestion would be to try to compute an Involutive Basis with respect to the left or right divisions to begin with (as they are easily defined and involutive reduction with respect to these divisions is very efficient); otherwise to try one of the 'overlap' divisions, choosing a strong overlap division if it is important to obtain disjoint involutive cones.

It is also worth mentioning that for all the divisions we have encountered so far, if Algorithm 12 terminates then it does so with a noncommutative Gröbner Basis, which means that Algorithm 12 can be thought of as an alternative algorithm for computing noncommutative Gröbner Bases. Whether this method is more or less efficient than computing noncommutative Gröbner Bases using Algorithm 5 is a matter for further discussion.

5.5.4 Alternative Divisions

Having encountered three different types of involutive division so far (each of which has two variants – left and right), let us now consider if there are any other involutive divisions with some useful properties, starting by thinking of global divisions.

Alternative Global Divisions

Open Question 2 Apart from the empty, left and right divisions, are there any other global involutive divisions of the following types:

(a) strong and continuous;

(b) weak, continuous and Gröbner?

Remark 5.5.26 It seems unlikely that a global division will exist that affirmatively answers Open Question 2 and does not either assign all variables to be left nonmultiplicative or all right nonmultiplicative (thus refining the right or left divisions respectively). The reason for saying this is because the moment you have one variable being left multiplicative and another variable being right multiplicative for the same monomial *globally*, then you risk not being able to prove that your division is strong; similarly the moment you have one variable being left nonmultiplicative and another variable being right nonmultiplicative for the same monomial globally, then you risk not being able to prove that your division is continuous.

Alternative Local Divisions

So far, all the local divisions we have considered have assigned all variables to be multiplicative on one side, and have chosen certain variables to be nonmultiplicative on the other side. Let us now consider a local division that modifies the left overlap division by assigning some variables to be nonmultiplicative on both left and right hand sides.

Definition 5.5.27 (The Two-Sided Left Overlap Division $\mathcal{W}$) Consider a set $U = \{u_1, \ldots, u_m\}$ of monomials, where all variables are assumed to be left and right multiplicative for all elements of U to begin with. Assign multiplicative variables to U according to Algorithm 16, which (in words) performs the following tasks.

(a) For all possible ways that a monomial $u_j \in U$ is a subword of a (different) monomial $u_i \in U$, so that
$$\text{Subword}(u_i, k, k + \deg(u_j) - 1) = u_j$$
for some integer k, assign the variable $\text{Subword}(u_i, k-1, k-1)$ to be left nonmultiplicative for u_j if u_j is a suffix of u_i; and assign the variable $\text{Subword}(u_i, k + \deg(u_j), k + \deg(u_j))$ to be right nonmultiplicative for u_j if u_j is not a suffix of u_i.

(b) For all possible ways that a proper prefix of a monomial $u_i \in U$ is equal to a proper suffix of a (not necessarily different) monomial $u_j \in U$, so that
$$\text{Prefix}(u_i, k) = \text{Suffix}(u_j, k)$$
for some integer k and u_i is not a subword of u_j or vice-versa, use the recipe provided in the second half of Algorithm 16 to ensure that at least one of the following conditions

are satisfied: (i) the variable $\text{Subword}(u_i, k+1, k+1)$ is right nonmultiplicative for u_j; (ii) the variable $\text{Subword}(u_j, \deg(u_j)-k, \deg(u_j)-k)$ is left nonmultiplicative for u_i.

Remark 5.5.28 For task (b) above, Algorithm 16 gives preference to monomials which are greater in the DegRevLex monomial ordering (given the choice, it always assigns a nonmultiplicative variable to whichever monomial out of u_i and u_j is the smallest); it also attempts to minimise the number of variables made nonmultiplicative by only assigning a variable to be nonmultiplicative if both the variables $\text{Subword}(u_i, k+1, k+1)$ and $\text{Subword}(u_j, \deg(u_j)-k, \deg(u_j)-k)$ are respectively right multiplicative and left multiplicative. These refinements will become crucial when proving the continuity of the division.

Example 5.5.29 Consider the set of monomials $U := \{zx^2yxy,\ yzx,\ xy\}$ over the polynomial ring $\mathbb{Q}\langle x, y, z\rangle$. Here are the left and right multiplicative variables for U with respect to the two-sided left overlap division $\mathcal{W}$.

u	$\mathcal{M}^L_{\mathcal{W}}(u, U)$	$\mathcal{M}^R_{\mathcal{W}}(u, U)$
zx^2yxy	$\{x, y, z\}$	$\{x, y, z\}$
yzx	$\{y, z\}$	$\{y, z\}$
xy	$\{x\}$	$\{y, z\}$

The above table is constructed from the table T shown below, a table which is obtained by applying Algorithm 16 to U.

Monomial	x^L	x^R	y^L	y^R	z^L	z^R
zx^2yxy	1	1	1	1	1	1
yzx	0	0	1	1	1	1
xy	1	0	0	1	0	1

The zero entries in T correspond to the following overlaps between the elements of U (presented in the order in which Algorithm 16 encounters them).

Algorithm 16 The Two-Sided Left Overlap Division $\mathcal{W}$

Input: A set of monomials $U = \{u_1, \ldots, u_m\}$ ordered by DegRevLex ($u_1 \geqslant u_2 \geqslant \cdots \geqslant u_m$), where $u_i \in R\langle x_1, \ldots, x_n \rangle$.

Output: A table T of left and right multiplicative variables for all $u_i \in U$, where each entry of T is either 1 (multiplicative) or 0 (nonmultiplicative).

Create a table T of multiplicative variables as shown below:

	x_1^L	x_1^R	x_2^L	x_2^R	$\cdots$	x_n^L	x_n^R
u_1	1	1	1	1	$\cdots$	1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
u_m	1	1	1	1	$\cdots$	1	1

for each monomial $u_i \in U$ $(1 \leqslant i \leqslant m)$ **do**
  for each monomial $u_j \in U$ $(i \leqslant j \leqslant m)$ **do**
    Let $u_i = x_{i_1}x_{i_2}\ldots x_{i_\alpha}$ and $u_j = x_{j_1}x_{j_2}\ldots x_{j_\beta}$;
    if $(i \neq j)$ **then**
      for each k $(1 \leqslant k \leqslant \alpha - \beta + 1)$ **do**
        if $(\text{Subword}(u_i, k, k + \beta - 1) == u_j)$ **then**
          if $(k < \alpha - \beta + 1)$ **then** $T(u_j, x^R_{i_{k+\beta}}) = 0$;
          else $T(u_j, x^L_{i_{k-1}}) = 0$;
          end if
        end if
      end for
    end if
    for each k $(1 \leqslant k \leqslant \beta - 1)$ **do**
      if $(\text{Prefix}(u_i, k) == \text{Suffix}(u_j, k))$ **then**
        if $(T(u_i, x^L_{j_{\beta-k}}) + T(u_j, x^R_{i_{k+1}}) == 2)$ **then** $T(u_j, x^R_{i_{k+1}}) = 0$;
        end if
      end if
      if $(\text{Suffix}(u_i, k) == \text{Prefix}(u_j, k))$ **then**
        if $(T(u_i, x^R_{j_{k+1}}) + T(u_j, x^L_{i_{\alpha-k}}) == 2)$ **then** $T(u_j, x^L_{i_{\alpha-k}}) = 0$;
        end if
      end if
    end for
  end for
end for
return T;

Table Entry	Overlap
$T(yzx, x^R)$	$\text{Prefix}(zx^2yxy, 2) = \text{Suffix}(yzx, 2)$
$T(yzx, x^L)$	$\text{Suffix}(zx^2yxy, 1) = \text{Prefix}(yzx, 1)$
$T(xy, x^R)$	$\text{Subword}(zx^2yxy, 3, 4) = xy$
$T(xy, y^L)$	$\text{Subword}(zx^2yxy, 5, 6) = xy$
$T(xy, z^L)$	$\text{Suffix}(yzx, 1) = \text{Prefix}(xy, 1)$

Notice that the overlap $\text{Prefix}(yzx, 1) = \text{Suffix}(xy, 1)$ does not produce a zero entry for $T(xy, z^R)$, as by the time that we encounter this overlap in the algorithm, we have already assigned $T(yzx, x^L) = 0$.

Proposition 5.5.30 *The two-sided left overlap division $\mathcal{W}$ is a weak involutive division.*

Proof: We refer to the proof of Proposition 5.5.15, making the obvious changes (for example replacing $\mathcal{O}$ by $\mathcal{W}$). $\Box$

For the following two propositions, we defer their proofs to Appendix A due to their length and technical nature.

Proposition 5.5.31 *The two-sided left overlap division $\mathcal{W}$ is continuous.*

Proof: We refer to Appendix A. $\Box$

Proposition 5.5.32 *The two-sided left overlap division $\mathcal{W}$ is a Gröbner involutive division.*

Proof: We refer to Appendix A, noting that the proof is similar to the proof of Proposition 5.5.16. $\Box$

Remark 5.5.33 Because a variable is sometimes only assigned nonmultiplicative if two other variables are multiplicative in Algorithm 16, the subset condition of Definition 5.1.6 will not always be satisfied with respect to the two-sided left overlap division. This will still hold true even if we apply Algorithm 14 at the end of Algorithm 16, which means that the two-sided left overlap division cannot be converted to give a strong involutive division in the same way that we converted the left overlap division to give the strong left overlap division.

To finish this section, let us now consider some further variations of the left overlap division, variations that will allow us to assign more multiplicative variables than the left overlap division (and hence potentially have to deal with fewer prolongations when using Algorithm 12), but variations that cannot be modified to give strong involutive divisions in the same way that the left overlap division was modified to give the strong left overlap division (this is because there are other ways beside a monomial being a suffix of another monomial that two involutive cones can be non-disjoint with respect to these modified divisions).

Definition 5.5.34 (The Prefix-Only Left Overlap Division) Let $U = \{u_1, \ldots, u_m\}$ be a set of monomials, and assume that all variables are left and right multiplicative for all elements of U to begin with.

(a) For all possible ways that a monomial $u_j \in U$ is a proper prefix of a monomial $u_i \in U$, assign the variable $\text{Subword}(u_i, \deg(u_j)+1, \deg(u_j)+1)$ to be right nonmultiplicative for u_j.

(b) For all possible ways that a proper prefix of a monomial $u_i \in U$ is equal to a proper suffix of a (not necessarily different) monomial $u_j \in U$, so that

$$\text{Prefix}(u_i, k) = \text{Suffix}(u_j, k)$$

for some integer k and u_i is not a subword of u_j or vice-versa, assign the variable $\text{Subword}(u_i, k+1, k+1)$ to be right nonmultiplicative for u_j.

Definition 5.5.35 (The Subword-Free Left Overlap Division) Consider a set $U = \{u_1, \ldots, u_m\}$ of monomials, where all variables are assumed to be left and right multiplicative for all elements of U to begin with.

For all possible ways that a proper prefix of a monomial $u_i \in U$ is equal to a proper suffix of a (not necessarily different) monomial $u_j \in U$, so that

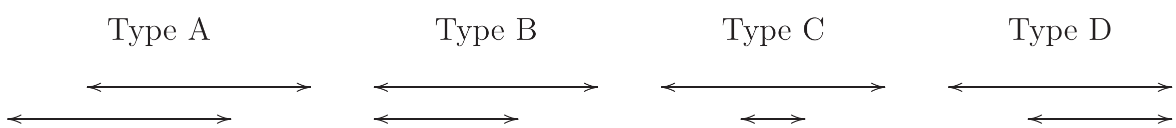
$$\text{Prefix}(u_i, k) = \text{Suffix}(u_j, k)$$

for some integer k and u_i is not a subword of u_j or vice-versa, assign the variable $\text{Subword}(u_i, k+1, k+1)$ to be right nonmultiplicative for u_j.

Proposition 5.5.36 *Both the prefix-only left overlap and the subword-free left overlap divisions are continuous, weak and Gröbner.*

Proof: We leave these proofs as exercises for the interested reader, noting that the proofs will be based on (and in some cases will be identical to) the proofs of Propositions 5.5.14, 5.5.15 and 5.5.16 respectively. □

Remark 5.5.37 To help distinguish between the different types of overlap division we have encountered in this chapter, let us now give the following table showing which types of overlap each overlap division considers.



Overlap Division Type	Overlap Type			
	A	B	C	D
Left	✓	✓	✓	×
Right	✓	×	✓	✓
Strong Left	✓	✓	✓	×
Strong Right	✓	×	✓	✓
Two-Sided Left	✓	✓	✓	✓
Two-Sided Right	✓	✓	✓	✓
Prefix-Only Left	✓	✓	×	×
Suffix-Only Right	✓	×	×	✓
Subword-Free Left	✓	×	×	×
Subword-Free Right	✓	×	×	×

5.6 Termination

Given a basis F generating an ideal over a noncommutative polynomial ring $\mathcal{R}$, does there exist a finite Involutive Basis for F with respect to some admissible monomial ordering O and some involutive division I? Unlike the commutative case, where the answer to the corresponding question (for certain divisions) is always 'Yes', the answer to this question can potentially be 'No', as if the noncommutative Gröbner Basis for F with respect to O is infinite, then the noncommutative Involutive Basis algorithm will not find a finite Involutive Basis for F with respect to I and O, as it will in effect be trying to compute the same infinite Gröbner Basis.

However, a valid follow-up question would be to ask whether the noncommutative Involutive Basis algorithm will terminate in the case that the noncommutative Gröbner Basis algorithm terminates. In Section 5.4, we defined a property of noncommutative involutive divisions (conclusivity) that ensures, when satisfied, that the answer to this secondary question is always 'Yes'. Despite this, we will not prove in this thesis that any of the divisions we have defined are conclusive. Instead, we leave the following open question for further investigation.

Open Question 3 Are there any conclusive noncommutative involutive divisions that are also continuous and either strong or Gröbner?

To obtain an affirmative answer to the above question, one approach may be to start by finding a proof for the following conjecture.

Conjecture 5.6.1 *Let O be an arbitrary admissible monomial ordering, and let I be an arbitrary involutive division that is continuous and either strong or Gröbner. When computing an Involutive Basis for some basis F with respect to O and I using Algorithm 12, if F possesses a finite unique reduced Gröbner Basis G with respect to O, then after a finite number of steps of Algorithm 12,* $\mathrm{LM}(G)$ *appears as a subset of the set of leading monomials of the current basis.*

To prove that a particular involutive division is conclusive, we would then need to show that once $\mathrm{LM}(G)$ appears as a subset of the set of leading monomials of the current basis, then the noncommutative Involutive Basis algorithm terminates (either immediately or in a finite number of steps), thus providing the required finite noncommutative Involutive Basis for F.

5.7 Examples

5.7.1 A Worked Example

Example 5.7.1 Let $F := \{f_1, f_2\} = \{x^2y^2 - 2xy^2 + x^2,\ x^2y - 2xy\}$ be a basis for an ideal J over the polynomial ring $\mathbb{Q}\langle x, y\rangle$, and let the monomial ordering be DegLex. Let us now compute a Locally Involutive Basis for F with respect to the strong left overlap division $\mathcal{S}$ and thick divisors using Algorithm 12.

To begin with, we must autoreduce the input set F. This leaves the set unchanged, as we can verify by using the following table of multiplicative variables (obtained by using Algorithm 15), where y is right nonmultiplicative for f_2 because of the overlap $\mathrm{LM}(f_2) = \mathrm{Subword}(\mathrm{LM}(f_1), 1, 3)$; and x is right nonmultiplicative for f_1 because we need to have a variable in $\mathrm{LM}(f_2)$ being right nonmultiplicative for f_1.

Polynomial	$\mathcal{M}^L_{\mathcal{S}}(f_i, F)$	$\mathcal{M}^R_{\mathcal{S}}(f_i, F)$
$f_1 = x^2y^2 - 2xy^2 + x^2$	$\{x, y\}$	$\{y\}$
$f_2 = x^2y - 2xy$	$\{x, y\}$	$\{x\}$

The above table also provides us with the set $S = \{f_1x, f_2y\} = \{x^2y^2x - 2xy^2x + x^3,\ x^2y^2 - 2xy^2\}$ of prolongations that is required for the next step of the algorithm. As $x^2y^2 < x^2y^2x$ in the DegLex monomial ordering, we involutively reduce the element $f_2y \in S$ first.

$$\begin{aligned} f_2y = x^2y^2 - 2xy^2 \quad &\xrightarrow[\mathcal{S}]{}_{f_1} \quad x^2y^2 - 2xy^2 - (x^2y^2 - 2xy^2 + x^2) \\ &= \quad -x^2. \end{aligned}$$

As the prolongation did not involutively reduce to zero, we now exit from the second while loop of Algorithm 12 and proceed by autoreducing the set $F \cup \{f_3 := -x^2\} = \{x^2y^2 - 2xy^2 + x^2,\ x^2y - 2xy,\ -x^2\}$.

Polynomial	$\mathcal{M}^L_{\mathcal{S}}(f_i, F)$	$\mathcal{M}^R_{\mathcal{S}}(f_i, F)$
$f_1 = x^2y^2 - 2xy^2 + x^2$	$\{x, y\}$	$\{y\}$
$f_2 = x^2y - 2xy$	$\{x, y\}$	$\emptyset$
$f_3 = -x^2$	$\{x, y\}$	$\emptyset$

This process involutively reduces the third term of f_1 using f_3, leaving the new set $\{f_4 := x^2y^2 - 2xy^2,\ f_2,\ f_3\}$ whose multiplicative variables are identical to the multiplicative variables of the set $\{f_1,\ f_2,\ f_3\}$ shown above.

Next, we construct the set $S = \{f_4x,\ f_2x,\ f_2y,\ f_3x,\ f_3y\}$ of prolongations, processing the element f_3y first.

$$\begin{aligned} f_3y = -x^2y \quad &\xrightarrow[\mathcal{S}]{}_{f_2} \quad -x^2y + (x^2y - 2xy) \\ &= \quad -2xy. \end{aligned}$$

Again the prolongation did not involutively reduce to zero, so we add the involutively reduced prolongation to our basis to obtain the set $\{f_4,\ f_2,\ f_3,\ f_5 := -2xy\}$.

Polynomial	$\mathcal{M}^L_{\mathcal{S}}(f_i, F)$	$\mathcal{M}^R_{\mathcal{S}}(f_i, F)$
$f_4 = x^2y^2 - 2xy^2$	$\{x, y\}$	$\{y\}$
$f_2 = x^2y - 2xy$	$\{x, y\}$	$\emptyset$
$f_3 = -x^2$	$\{x, y\}$	$\emptyset$
$f_5 = -2xy$	$\{x, y\}$	$\emptyset$

This time during autoreduction, the polynomial f_2 involutively reduces to zero with respect to the set $\{f_4,\ f_3,\ f_5\}$:

$$\begin{aligned} f_2 = x^2y - 2xy \quad & \xrightarrow[\mathcal{S}]{}_{f_5} \quad x^2y - 2xy + \frac{1}{2}x(-2xy) \\ & = \quad -2xy \\ & \xrightarrow[\mathcal{S}]{}_{f_5} \quad -2xy - (-2xy) \\ & = \quad 0. \end{aligned}$$

This leaves us with the set $\{f_4,\ f_3,\ f_5\}$ after autoreduction is complete.

Polynomial	$\mathcal{M}^L_{\mathcal{S}}(f_i, F)$	$\mathcal{M}^R_{\mathcal{S}}(f_i, F)$
$f_4 = x^2y^2 - 2xy^2$	$\{x, y\}$	$\{y\}$
$f_3 = -x^2$	$\{x, y\}$	$\emptyset$
$f_5 = -2xy$	$\{x, y\}$	$\emptyset$

The next step is to construct the set $S = \{f_4x, f_3x, f_3y, f_5x, f_5y\}$ of prolongations, from which the element f_5y is processed first.

$$f_5y = -2xy^2 =: f_6.$$

When the set $\{f_4,\ f_3,\ f_5,\ f_6\}$ is autoreduced, the polynomial f_4 now involutively reduces to zero, leaving us with the autoreduced set $\{f_3,\ f_5,\ f_6\} = \{-x^2,\ -2xy,\ -2xy^2\}$.

Polynomial	$\mathcal{M}^L_{\mathcal{S}}(f_i, F)$	$\mathcal{M}^R_{\mathcal{S}}(f_i, F)$
$f_3 = -x^2$	$\{x, y\}$	$\emptyset$
$f_5 = -2xy$	$\{x, y\}$	$\emptyset$
$f_6 = -2xy^2$	$\{x, y\}$	$\{y\}$

Our next task is to process the elements of the set $S = \{f_3x,\ f_3y,\ f_5x,\ f_5y,\ f_6x\}$ of prolongations. The first element f_5y we pick from S involutively reduces to zero, but the second element f_5x does not:

$$\begin{aligned} f_5y = -2xy^2 \quad &\xrightarrow[\mathcal{S}]{} {}_{f_6} \quad -2xy^2 - (-2xy^2) \\ &= \quad 0; \end{aligned}$$

$$f_5x = -2xyx \quad =: \quad f_7.$$

After constructing the set $\{f_3,\ f_5,\ f_6,\ f_7\}$, autoreduction does not alter the contents of the set, leaving us to construct our next set of prolongations from the following table of multiplicative variables.

Polynomial	$\mathcal{M}^L_{\mathcal{S}}(f_i, F)$	$\mathcal{M}^R_{\mathcal{S}}(f_i, F)$
$f_3 = -x^2$	$\{x, y\}$	$\emptyset$
$f_5 = -2xy$	$\{x, y\}$	$\emptyset$
$f_6 = -2xy^2$	$\{x, y\}$	$\{y\}$
$f_7 = -2xyx$	$\{x, y\}$	$\emptyset$

Whilst processing this (7 element) set of prolongations, we add the involutively irreducible prolongation $f_6x = -2xy^2x =: f_8$ to our basis to give a five element set which in unaffected by autoreduction.

Polynomial	$\mathcal{M}^L_{\mathcal{S}}(f_i, F)$	$\mathcal{M}^R_{\mathcal{S}}(f_i, F)$
$f_3 = -x^2$	$\{x, y\}$	$\emptyset$
$f_5 = -2xy$	$\{x, y\}$	$\emptyset$
$f_6 = -2xy^2$	$\{x, y\}$	$\{y\}$
$f_7 = -2xyx$	$\{x, y\}$	$\emptyset$
$f_8 = -2xy^2x$	$\{x, y\}$	$\emptyset$

To finish, we analyse the elements of the set

$$S = \{f_3x,\ f_3y,\ f_5x,\ f_5y,\ f_6x,\ f_7x,\ f_7y,\ f_8x,\ f_8y\}$$

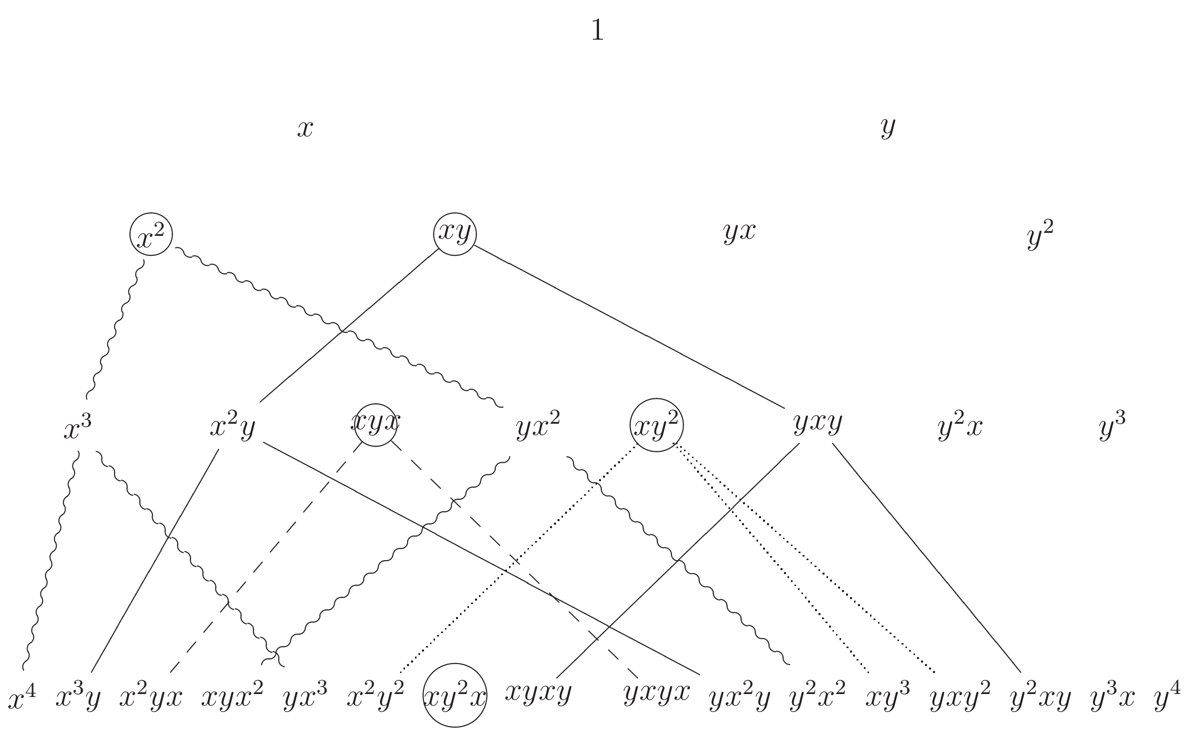
of prolongations in the order f_5y, f_5x, f_3y, f_3x, f_6x, f_7y, f_7x, f_8y, f_8x.

$$\begin{aligned} f_5y = -2xy^2 & \xrightarrow[\mathcal{O}]{} {}_{f_6} & -2xy^2 - (-2xy^2) \\ & = & 0; \\ & \vdots & \\ f_8x = -2xy^2x^2 & \xrightarrow[\mathcal{O}]{} {}_{f_3} & -2xy^2x^2 - 2xy^2(-x^2) \\ & = & 0. \end{aligned}$$

Because all prolongations involutively reduce to zero (and hence $S = \emptyset$), the algorithm now terminates with the Involutive Basis

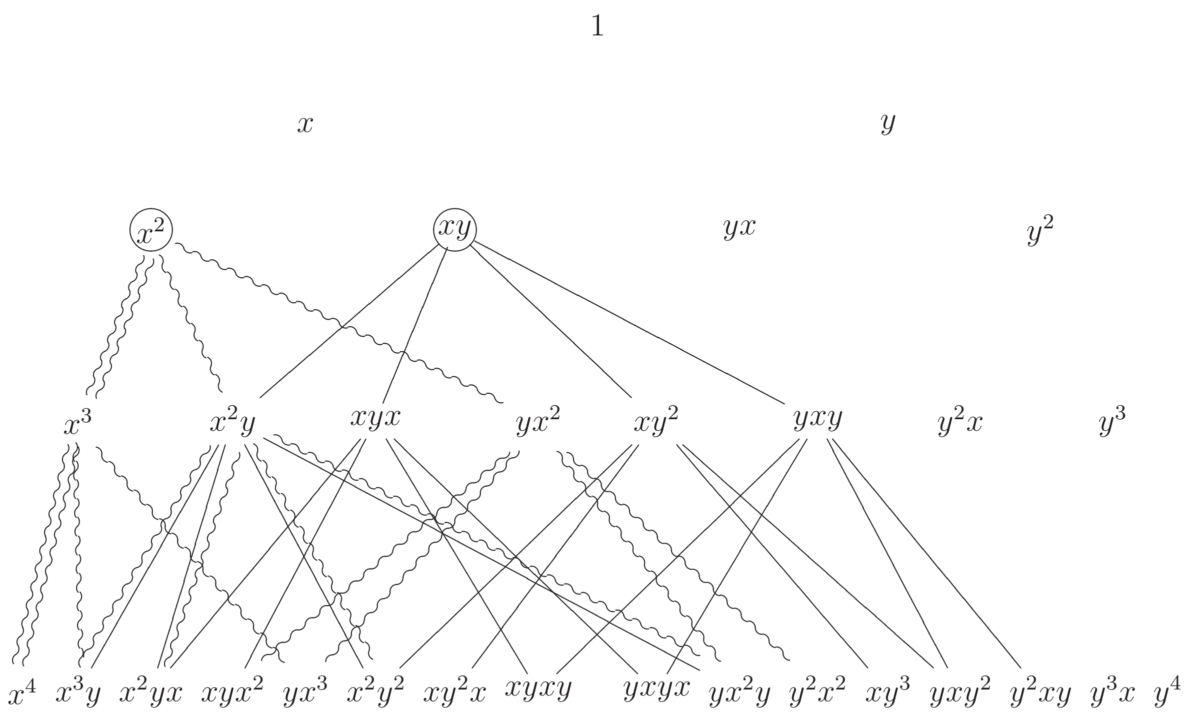
$$G := \{-x^2,\ -2xy,\ -2xy^2,\ -2xyx,\ -2xy^2x\}$$

as output, a basis which can be visualised by looking at the following (partial) involutive monomial lattice for G.



For comparison, the (partial) monomial lattice of the reduced DegLex Gröbner Basis H

for F is shown below, where $H := \{x^2, xy\}$ is obtained by applying Algorithm 6 to G.



Looking at the lattices, we can verify that the involutive cones give a disjoint cover of the conventional cones *up to monomials of degree 4*. However, if we were to draw the next part of the lattices (monomials of degree 5), we would notice that the monomial xy^3x is conventionally reducible by the Gröbner Basis, but is not involutively reducible by the Involutive Basis. This fact verifies that when thick divisors are being used, a Locally Involutive Basis is not necessarily an Involutive Basis, as for G to be an Involutive Basis with respect to $\mathcal{S}$ and thick divisors, the monomial xy^3x has to be involutively reducible with respect to G.

5.7.2 Involutive Rewrite Systems

Remark 5.7.2 In this section, we use terminology from the theory of term rewriting that has not yet been introduced in this thesis. For an elementary introduction to this theory, we refer to [5], [19] and [36].

Let $C = \langle A \mid B \rangle$ be a monoid rewrite system, where $A = \{a_1, \ldots, a_n\}$ is an alphabet and $B = \{b_1, \ldots, b_m\}$ is a set of rewrite rules of the form $b_i = \ell_i \to r_i$ ($1 \leqslant i \leqslant m$; $\ell_i, r_i \in A^*$). Given a fixed admissible well-order on the words in A compatible with C, the

Knuth-Bendix critical pairs completion algorithm [39] attempts to find a complete rewrite system C' for C that is Noetherian and confluent, so that any word over the alphabet A has a unique normal form with respect to C'. The algorithm proceeds by considering overlaps of left hand sides of rules, forming new rules when two reductions of an overlap word result in two distinct normal forms.

It is well known (see for example [33]) that the Knuth-Bendix critical pairs completion algorithm is a special case of the noncommutative Gröbner Basis algorithm. To find a complete rewrite system for C using Algorithm 5, we treat C as a set of polynomials $F = \{\ell_1 - r_1,\, \ell_2 - r_2,\, \ldots,\, \ell_m - r_m\}$ generating a two-sided ideal over the noncommutative polynomial ring $\mathbb{Z}\langle a_1, \ldots, a_n\rangle$, and we compute a noncommutative Gröbner Basis G for F using a monomial ordering induced from the fixed admissible well-order on the words in A.

Because every noncommutative Involutive Basis (with respect to a strong or Gröbner involutive division) is a noncommutative Gröbner Basis, it is clear that a complete rewrite system for C can now also be obtained by computing an Involutive Basis for F, a complete rewrite system we shall call an *involutive complete rewrite system*.

The advantage of involutive complete rewrite systems over conventional complete rewrite systems is that the unique normal form of any word over the alphabet A can be obtained uniquely with respect to an involutive complete rewrite system (subject of course to certain conditions (such as working with a strong involutive division) being satisfied), a fact that will be illustrated in the following example.

Example 5.7.3 Let $C := \langle Y, X, y, x \mid x^3 \to \varepsilon,\, y^2 \to \varepsilon,\, (xy)^2 \to \varepsilon,\, Xx \to \varepsilon,\, xX \to \varepsilon,\, Yy \to \varepsilon,\, yY \to \varepsilon\rangle$ be a monoid rewrite system for the group S_3, where ε denotes the empty word, and $Y > X > y > x$ is the alphabet ordering. If we apply the Knuth-Bendix algorithm to C with respect to the DegLex (word) ordering, we obtain the complete rewrite system

$$C' := \langle Y, X, y, x \mid xyx \to y,\ yxy \to X,\ x^2 \to X,\ Xx \to \varepsilon,\ y^2 \to \varepsilon,\ Xy \to yx,\ xX \to \varepsilon,\ yX \to xy,\ X^2 \to x,\ Y \to y\rangle.$$

With respect to the DegLex monomial ordering and the left division, if we apply Algorithm 12 to the basis $F := \{x^3-1,\ y^2-1,\ (xy)^2-1,\ Xx-1,\ xX-1,\ Yy-1,\ yY-1\}$ corresponding to C, we obtain the following Involutive Basis for F (which we have converted back to a

rewrite system to give an involutive complete rewrite system C'' for C).

$$C'' := \langle Y, X, y, x \mid y^2 \to \varepsilon,\ Xx \to \varepsilon,\ xX \to \varepsilon,\ Yy \to \varepsilon,\ y^2x \to x,\ Y \to y,\ Yx \to yx,\ Xxy \to y,\ Yyx \to x,\ x^2 \to X,\ X^2 \to x,\ xyx \to y,\ Xy \to yx,\ Xyx \to xy,\ x^2y \to yx,\ yX \to xy,\ yxy \to X,\ Yxy \to X,\ YX \to xy \rangle.$$

With the involutive complete rewrite system, we are now able to uniquely reduce each word over the alphabet $\{Y, X, y, x\}$ to one of the six elements of S_3. To illustrate this, consider the word $yXYx$. Using the 10 element complete rewrite system C' obtained by using the Knuth-Bendix algorithm, there are several reduction paths for this word, as illustrated by the following diagram.

$$\begin{tikzcd}
 & yXYx \arrow[dl, "yX\to xy"'] \arrow[dr, "Y\to y"] & & & & \\
xyYx \arrow[dr, "Y\to y"'] & & yXyx \arrow[dl, "yX\to xy"] \arrow[dr, "Xy\to yx"] & & & \\
 & xy^2x \arrow[dr, "y^2\to\varepsilon"'] & & y^2x^2 \arrow[dl, "y^2\to\varepsilon"'] \arrow[dr, "x^2\to X"] & & \\
 & & x^2 \arrow[dr, "x^2\to X"'] & & y^2X \arrow[dl, "y^2\to\varepsilon"] \arrow[dr, "yX\to xy"] & \\
 & & & X & & yxy \arrow[d, "yxy\to X"] \\
 & & & & & X
\end{tikzcd}$$

However, by involutively reducing the word $yXYx$ with respect to the 19 element involutive complete rewrite system C'', there is only one reduction path, namely

$$\begin{tikzcd}
yXYx \arrow[d, "Yx\to yx"] \\
yXyx \arrow[d, "Xyx\to xy"] \\
yxy \arrow[d, "yxy\to X"] \\
X
\end{tikzcd}$$

5.7.3 Comparison of Divisions

Following on from the S_3 example above, consider the basis $F := \{x^4 - 1,\ y^3 - 1,\ (xy)^2 - 1,\ Xx - 1,\ xX - 1,\ Yy - 1,\ yY - 1\}$ over the polynomial ring $\mathbb{Q}\langle Y, X, y, x\rangle$ corresponding to a monoid rewrite system for the group S_4. With the monomial ordering being DegLex, below we present some data collected when, whilst using a prototype implementation of Algorithm 12 (as given in Appendix B), an Involutive Basis is computed for F with respect to several different involutive divisions (the reduced DegLex Gröbner Basis for F has 21 elements).

Remark 5.7.4 The program was run using FreeBSD 5.4 on an AMD Athlon XP 1800+ with 512MB of memory.

Key	Involutive Division	Key	Involutive Division
1	Left Division	7	Subword-Free Left Overlap Division
2	Right Division	8	Right Overlap Division
3	Left Overlap Division	9	Strong Right Overlap Division
4	Strong Left Overlap Division	10	Two-Sided Right Overlap Division
5	Two-Sided Left Overlap Division	11	Suffix-Only Right Overlap Division
6	Prefix-Only Left Overlap Division	12	Subword-Free Right Overlap Division

Division	Size of Basis	Number of Prolongations	Number of Involutive Reductions	Time
1	73	104	15947	0.77
2	73	104	13874	0.74
3	65	64	10980	8.62
4	73	94	15226	23.14
5	77	70	12827	16.04
6	65	64	10980	8.97
7	65	64	10980	7.13
8	73	76	11046	13.27
9	73	95	13240	26.16
10	87	80	13005	24.53
11	73	76	11046	13.40
12	69	82	10458	9.52

We note that the algorithm completes quickest with respect to the global left or right divisions, as (i) we can take advantage of the efficient involutive reduction with respect to these divisions (see Section 5.5.1); and (ii) the multiplicative variables for a particular monomial with respect to these divisions is fixed (each time the basis changes when using one of the other local divisions, the multiplicative variables have to be recomputed). However, we also note that more prolongations are needed when using the left or right divisions, so that, in the long run, if we can devise an efficient way of finding the multiplicative variables for a set of monomials with respect to one of the local divisions, then the algorithm could (perhaps) terminate more quickly than for the two global divisions.

5.8 Improvements to the Noncommutative Involutive Basis Algorithm

5.8.1 Efficient Reduction

Conventionally, we divide a noncommutative polynomial p with respect to a set of polynomials P using Algorithm 2. In this algorithm, an important step is to find out if a polynomial in P divides one of the monomials u in the polynomial we are currently reducing, stated as the condition '**if** $(\mathrm{LM}(p_j) \mid u)$ **then**' in Algorithm 2. One way of finding out if this condition is satisfied would be to execute the following piece of code, where $\alpha := \deg(u)$; $\beta := \deg(\mathrm{LM}(p_j))$; and we note that $\alpha - \beta + 1$ operations are potentially needed to find out if the condition is satisfied.

$i = 1$;
while $(i \leqslant \alpha - \beta + 1)$ **do**
  if $(\mathrm{LM}(p_j) == \mathrm{Subword}(u, i, i + \beta - 1))$ **then**
    return true;
  else
    $i = i + 1$;
  end if
end while
return false;

When involutively dividing a polynomial p with respect to a set of polynomials P and some involutive division I, the corresponding problem is to find out if some monomial

$\mathrm{LM}(p_j)$ is an *involutive* divisor of some monomial u. At first glance, this problem seems more difficult than the problem of finding out if $\mathrm{LM}(p_j)$ is a conventional divisor of u, as it is not just sufficient to discover one way that $\mathrm{LM}(p_j)$ divides u (as in the code above) — we have to verify that if we find a conventional divisor of u, then it is also an involutive divisor of u. Naively, assuming that thin divisors are being used, we could solve the problem using the code shown below, code that is clearly less efficient than the code for the conventional case shown above.

$i = 1$;
while $(i \leqslant \alpha - \beta + 1)$ **do**
  if $(\mathrm{LM}(p_j) == \mathrm{Subword}(u, i, i + \beta - 1))$ **then**
    if $((i == 1)$ **or** $((i > 1)$ **and** $(\mathrm{Subword}(u, i - 1, i - 1) \in \mathcal{M}_I^L(\mathrm{LM}(p_j), \mathrm{LM}(P))))$
    then
      if $((i == \alpha - \beta + 1)$ **or** $((i < \alpha - \beta + 1)$ **and** $(\mathrm{Subword}(u, i + \beta, i + \beta) \in \mathcal{M}_I^R(\mathrm{LM}(p_j), \mathrm{LM}(P))))$ **then**
        return true;
      end if
    end if
  else
    $i = i + 1$;
  end if
end while
return false;

However, for certain involutive divisions, it is possible to take advantage of some of the properties of these divisions in order to make it easier to discover whether $\mathrm{LM}(p_j)$ is an involutive divisor of u. We have already seen this in action in Section 5.5.1, where we saw that $\mathrm{LM}(p_j)$ can only involutively divide u with respect to the left or right divisions if $\mathrm{LM}(p_j)$ is a suffix or prefix of u respectively.

Let us now consider an improvement to be used whenever (i) an 'overlap' division that assigns all variables to be either left multiplicative or right multiplicative is used (ruling out any 'two-sided' overlap divisions); and (ii) thick divisors are being used. For the case of such an overlap division that assigns all variables to be left multiplicative (for example the left overlap division), the following piece of code can be used to discover whether or not $\mathrm{LM}(p_j)$ is an involutive divisor of u (note that a similar piece of code can be given for the case of an overlap division assigning all variables to be right multiplicative).

```
k = α; skip = 0;
while (k ⩾ β + 1) do
  if (Subword(u, k, k) ∉ M_I^R(LM(p_j), LM(P))) then
    skip = k; k = β;
  else
    k = k − 1;
  end if
end while

if (skip == 0) then
  i = 1;
else
  i = skip − β + 1;
end if

while (i ⩽ α − β + 1) do
  if (LM(p_j) == Subword(u, i, i + β − 1)) then
    return true;
  else
    i = i + 1;
  end if
end while
return false;
```

We note that the final section of the code (from '**while** $(i \leqslant \alpha - \beta + 1)$ **do**' onwards) is identical to the code for conventional reduction; the code before this just chooses the initial value of i (we rule out looking at certain subwords by analysing which variables in u are right nonmultiplicative for $\mathrm{LM}(p_j)$). For example, if $u := xy^2xyxy$; $\mathrm{LM}(p_j) := xyx$; and only the variable x is right nonmultiplicative for p_j, then in the conventional case we need 4 subword comparisons before we discover that $\mathrm{LM}(p_j)$ is a conventional divisor of u; but in the involutive case (using the code shown above) we only need 1 subword comparison before we discover that $\mathrm{LM}(p_j)$ is an involutive divisor of u (this is because the variable $\mathrm{Subword}(u, 6, 6) = x$ is right nonmultiplicative for $\mathrm{LM}(p_j)$, leaving just two subwords of u that are potentially equal to $\mathrm{LM}(p_j)$ in such a way that $\mathrm{LM}(p_j)$ is an involutive divisor of u).

Conventional Reduction

	x	y	y	x	y	x	y
$i = 1$	x	y	x				
$i = 2$		x	y	x			
$i = 3$			x	y	x		
$i = 4$				x	y	x	

Involutive Reduction

	x	y	y	x	y	x	y
$i = 4$				x	y	x	

Of course our new algorithm will not always 'win' in every case (for example if $u := xyx^2yxy$ and $\mathrm{LM}(p_j) := xyx$), and we will always have the overhead from having to determine the initial value of i, but the impression should be that we have more freedom in the involutive case to try these sorts of tricks, tricks which may lead to involutive reduction being more efficient than conventional reduction.

5.8.2 Improved Algorithms

Just as Algorithm 9 was generalised to give an algorithm for computing noncommutative Involutive Bases in Algorithm 12, it is conceivable that other algorithms for computing commutative Involutive Bases (as seen for example in [24]) can be generalised to the noncommutative case. Indeed, in the source code given in Appendix B, a noncommutative version of an algorithm found in [23, Section 5] for computing commutative Involutive Bases is given; we present below data obtained by applying this new algorithm to our S_4 example from Section 5.7.3 (the data from Section 5.7.3 is given in brackets for comparison; we see that the new algorithm generally analyses more prolongations but performs less involutive reduction).

Division	Size of Basis	Number of Prolongations	Number of Involutive Reductions	Time
1	73 (73)	323 (104)	875 (15947)	0.72 (0.77)
2	73 (73)	327 (104)	929 (13874)	0.83 (0.74)
3	70 (65)	288 (64)	831 (10980)	5.94 (8.62)
4	73 (73)	318 (94)	863 (15226)	4.62 (23.14)
5	70 (77)	288 (70)	831 (12827)	5.79 (16.04)
6	70 (65)	288 (64)	831 (10980)	5.71 (8.97)
7	69 (65)	288 (64)	833 (10980)	5.33 (7.13)
8	68 (73)	358 (76)	1092 (11046)	28.51 (13.27)
9	73 (73)	322 (95)	917 (13240)	6.39 (26.16)
10	68 (87)	358 (80)	1092 (13005)	28.75 (24.53)
11	68 (73)	358 (76)	1092 (11046)	28.54 (13.40)
12	66 (69)	364 (82)	1127 (10458)	28.87 (9.52)

5.8.3 Logged Involutive Bases

A (noncommutative) Logged Involutive Basis expresses each member of an Involutive Basis in terms of members of the original basis from which the Involutive Basis was computed.

Definition 5.8.1 Let $G = \{g_1, \ldots, g_p\}$ be an Involutive Basis computed from an initial basis $F = \{f_1, \ldots, f_m\}$. We say that G is a *Logged Involutive Basis* if, for each $g_i \in G$, we have an explicit expression of the form

$$g_i = \sum_{\alpha=1}^{\beta} \ell_\alpha f_{k_\alpha} r_\alpha,$$

where the ℓ_α and the r_α are terms and $f_{k_\alpha} \in F$ for all $1 \leqslant \alpha \leqslant \beta$.

Proposition 5.8.2 *Let $F = \{f_1, \ldots, f_m\}$ be a finite basis over a noncommutative polynomial ring. If we can compute an Involutive Basis for F, then it is always possible to compute a Logged Involutive Basis for F.*

Proof: Let $G = \{g_1, \ldots, g_p\}$ be an Involutive Basis computed from the initial basis $F = \{f_1, \ldots, f_m\}$ using Algorithm 12 (where $f_i \in R\langle x_1, \ldots, x_n\rangle$ for all $f_i \in F$). If an arbitrary $g_i \in G$ is not a member of the original basis F, then either g_i is an involutively

reduced prolongation, or g_i is obtained through the process of autoreduction. In the former case, we can express g_i in terms of members of F by substitution because either

$$g_i = x_j h - \sum_{\alpha=1}^{\beta} \ell_\alpha h_{k_\alpha} r_\alpha$$

or

$$g_i = h x_j - \sum_{\alpha=1}^{\beta} \ell_\alpha h_{k_\alpha} r_\alpha$$

for a variable x_j; terms ℓ_α and r_α; and polynomials h and h_{k_α} which we already know how to express in terms of members of F. In the latter case,

$$g_i = h - \sum_{\alpha=1}^{\beta} \ell_\alpha h_{k_\alpha} r_\alpha$$

for terms ℓ_α, r_α and polynomials h and h_{k_α} which we already know how to express in terms of members of F, so it follows that we can again express g_i in terms of members of F. $\square$

Example 5.8.3 Let $F := \{f_1, f_2\} = \{x^3 + 3xy - yx,\ y^2 + x\}$ generate an ideal over the polynomial ring $\mathbb{Q}\langle x, y\rangle$; let the monomial ordering be DegLex; and let the involutive division be the left division. In obtaining an Involutive Basis for F using Algorithm 12, a polynomial is added to F; f_1 is involutively reduced during autoreduction; and then four more polynomials are added to F, giving an Involutive Basis $G := \{g_1, g_2, g_3, g_4, g_5, g_6, g_7\} = \{x^3 + 2yx,\ y^2 + x,\ xy - yx,\ y^2x + x^2,\ xyx - yx^2,\ y^2x^2 - 2yx,\ xyx^2 - 2x^2\}$.

The five new polynomials were obtained by involutively reducing the prolongations $f_2 y$,

f_2x, g_3x, g_4x and g_5x respectively.

$$\begin{array}{rcl}
f_2y & = & y^3 + xy \\
 & \xrightarrow[\lhd]{}_{f_2} & y^3 + xy - y(y^2 + x) \\
 & = & xy - yx; \\
f_2x & = & y^2x + x^2; \\
g_3x & = & xyx - yx^2; \\
g_4x & = & y^2x^2 + x^3 \\
 & \xrightarrow[\lhd]{}_{g_1} & y^2x^2 + x^3 - (x^3 + 2yx) \\
 & = & y^2x^2 - 2yx; \\
g_5x & = & xyx^2 - yx^3 \\
 & \xrightarrow[\lhd]{}_{g_1} & xyx^2 - yx^3 + y(x^3 + 2yx) \\
 & = & xyx^2 + 2y^2x \\
 & \xrightarrow[\lhd]{}_{g_4} & xyx^2 + 2y^2x - 2(y^2x + x^2) \\
 & = & xyx^2 - 2x^2.
\end{array}$$

These reductions (plus the reduction

$$\begin{array}{rcl}
f_1 & \xrightarrow[\lhd]{}_{g_3} & x^3 + 3xy - yx - 3(xy - yx) \\
 & = & x^3 + 2yx
\end{array}$$

of f_1 performed during autoreduction after g_3 is added to F) enable us to give the following Logged Involutive Basis for F.

Member of G	Logged Representation
$g_1 = x^3 + 2yx$	$f_1 - 3f_2y + 3yf_2$
$g_2 = y^2 + x$	f_2
$g_3 = xy - yx$	$f_2y - yf_2$
$g_4 = y^2x + x^2$	f_2x
$g_5 = xyx - yx^2$	$f_2yx - yf_2x$
$g_6 = y^2x^2 - 2yx$	$-f_1 + f_2x^2 + 3f_2y - 3yf_2$
$g_7 = xyx^2 - 2x^2$	$yf_1 + 3y^2f_2 + f_2yx^2 - 2f_2x - yf_2x^2 - 3yf_2y$

Chapter 6

Gröbner Walks

When computing any Gröbner or Involutive Basis, the monomial ordering that has been chosen is a major factor in how long it will take for the algorithm to complete. For example, consider the ideal J generated by the basis $F := \{-2x^3z + y^4 + y^3z - x^3 + x^2y,\ 2xy^2z + yz^3 + 2yz^2,\ x^3y + 2yz^3 - 3yz^2 + 2\}$ over the polynomial ring $\mathbb{Q}[x, y, z]$. Using our test implementation of Algorithm 3, it takes less than a tenth of a second to compute a Gröbner Basis for F with respect to the DegRevLex monomial ordering, but 90 seconds to compute a Gröbner Basis for F with respect to Lex.

The Gröbner Walk, introduced by Collart, Kalkbrener and Mall in [18], forms part of a family of basis conversion algorithms that can convert Gröbner Bases with respect to 'fast' monomial orderings to Gröbner Bases with respect to 'slow' monomial orderings (see Section 2.5.4 for a brief discussion of other basis conversion algorithms). This process is often quicker than computing a Gröbner Basis for the 'slow' monomial ordering directly, as can be demonstrated by stating that in our test implementation of the Gröbner Walk, it only takes half a second to compute a Lex Gröbner Basis for the basis F defined above.

In this chapter, we will first recall the theory of the (commutative) Gröbner Walk, based on [18] and a paper [1] by Amrhein, Gloor and Küchlin; the reader is encouraged to read these papers in conjunction with this Chapter. We then describe two generalisations of the theory to give (i) a commutative Involutive Walk (due to Golubitsky [30]); and (ii) noncommutative Walks between harmonious monomial orderings.

6.1 Commutative Walks

To convert a Gröbner Basis with respect to one monomial ordering to a Gröbner Basis with respect to another monomial ordering, the Gröbner Walk works with the matrices associated to the orderings. Fortunately, [48] and [56] assert that any commutative monomial ordering has an associated matrix, allowing the Gröbner Walk to convert between any two monomial orderings.

6.1.1 Matrix Orderings

Definition 6.1.1 Let m be a monomial over a polynomial ring $R[x_1, \dots, x_n]$ with associated multidegree $(e^1, \dots, e^n)$. If $\omega = (\omega^1, \dots, \omega^n)$ is an n-dimensional weight vector (where $\omega^i \in \mathbb{Q}$ for all $1 \leqslant i \leqslant n$), we define the *$\omega$-degree of m*, written $\deg_\omega(m)$, to be the value

$$\deg_\omega(m) = (e^1 \times \omega^1) + (e^2 \times \omega^2) + \cdots + (e^n \times \omega^n).$$

Remark 6.1.2 The ω-degree of any term is equal to the ω-degree of the term's associated monomial.

Definition 6.1.3 Let m_1 and m_2 be two monomials over a polynomial ring $R[x_1, \dots, x_n]$ with associated multidegrees $e_1 = (e_1^1, \dots, e_1^n)$ and $e_2 = (e_2^1, \dots, e_2^n)$; and let Ω be an $N \times n$ matrix. If ω_i denotes the n-dimensional weight vector corresponding to the i-th row of Ω, then Ω determines a monomial ordering as follows: $m_1 < m_2$ if $\deg_{\omega_i}(m_1) < \deg_{\omega_i}(m_2)$ for some $1 \leqslant i \leqslant N$ and $\deg_{\omega_j}(m_1) = \deg_{\omega_j}(m_2)$ for all $1 \leqslant j < i$.

Definition 6.1.4 The corresponding matrices for the five monomial orderings defined in Section 1.2.1 are

$$\text{Lex} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}; \quad \text{InvLex} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix};$$

$$\text{DegLex} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}; \quad \text{DegInvLex} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \end{pmatrix};$$

$$\text{DegRevLex} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & -1 \\ 0 & 0 & 0 & \dots & -1 & 0 \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 0 & -1 & \dots & 0 & 0 \\ 0 & -1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Example 6.1.5 Let $m_1 := x^2y^2z^2$ and $m_2 := x^2y^3z$ be two monomials over the polynomial ring $\mathcal{R} := \mathbb{Q}[x, y, z]$. According to the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

representing the DegLex monomial ordering with respect to $\mathcal{R}$, we can deduce that $m_1 < m_2$ because $\deg_{\omega_1}(m_1) = \deg_{\omega_1}(m_2) = 6$; $\deg_{\omega_2}(m_1) = \deg_{\omega_2}(m_2) = 2$; and $\deg_{\omega_3}(m_1) = 2 < \deg_{\omega_3}(m_2) = 3$.

Definition 6.1.6 Given a polynomial p and a weight vector ω, the *initial* of p with respect to ω, written $\text{in}_\omega(p)$, is the sum of those terms in p that have maximal ω-degree. For example, if $\omega = (0, 1, 1)$ and $p = x^4 + xy^2z + y^3 + xz^2$, then $\text{in}_\omega(p) = xy^2z + y^3$.

Definition 6.1.7 A weight vector ω is *compatible* with a monomial ordering O if, given any polynomial $p = t_1 + \dots + t_m$ ordered in descending order with respect to O, $\deg_\omega(t_1) \geqslant \deg_\omega(t_i)$ holds for all $1 < i \leqslant m$.

6.1.2 The Commutative Gröbner Walk Algorithm

We present in Algorithm 17 an algorithm to perform the Gröbner Walk, modified from an algorithm given in [1].

Algorithm 17 The Commutative Gröbner Walk Algorithm

Input: A Gröbner Basis $G = \{g_1, g_2, \ldots, g_m\}$ with respect to an admissible monomial ordering O with an associated matrix A, where G generates an ideal J over a commutative polynomial ring $\mathcal{R} = R[x_1, \ldots, x_n]$.

Output: A Gröbner Basis $H = \{h_1, h_2, \ldots, h_p\}$ for J with respect to an admissible monomial ordering O' with an associated matrix B.

Let ω and τ be the weight vectors corresponding to the first rows of A and B;
Let C be the matrix whose first row is equal to ω and whose remainder is equal to the whole of the matrix B;
$t = 0$; found = false;
repeat
  Let $G' = \{\text{in}_\omega(g_i)\}$ for all $g_i \in G$;
  Compute a reduced Gröbner Basis H' for G' with respect to the monomial ordering defined by the matrix C (use Algorithms 3 and 4);
  $H = \emptyset$;
  for each $h' \in H'$ **do**
    Let $\sum_{i=1}^{j} p_i g'_i$ be the logged representation of h' with respect to G' (where $g'_i \in G'$ and $p_i \in \mathcal{R}$), obtained either through computing a Logged Gröbner Basis above or by dividing h' with respect to G';
    $H = H \cup \{\sum_{i=1}^{j} p_i g_i\}$, where $\text{in}_\omega(g_i) = g'_i$;
  end for
  Reduce H with respect to C (use Algorithm 4);
  if $(t == 1)$ **then**
    found = true;
  else
    $t = \min(\{s \mid \deg_{\omega(s)}(p_1) = \deg_{\omega(s)}(p_i), \deg_{\omega(0)}(p_1) \neq \deg_{\omega(0)}(p_i),$
    $h = p_1 + \cdots + p_k \in H\} \cap (0, 1])$, where $\omega(s) := \omega + s(\tau - \omega)$ for $0 \leqslant s \leqslant 1$;
  end if
  if (t is undefined) **then**
    found = true;
  else
    $G = H$; $\omega = (1 - t)\omega + t\tau$;
  end if
until (found = true)
return H;

Some Remarks:

- In the first iteration of the **repeat** ... **until** loop, G' is a Gröbner Basis for the ideal[1] $\text{in}_\omega(J)$ with respect to the monomial ordering defined by C, as ω is compatible with C. During subsequent iterations of the same loop, G' is a Gröbner Basis for the ideal $\text{in}_\omega(J)$ with respect to the monomial ordering used to compute H during the previous iteration of the **repeat** ... **until** loop, as ω is compatible with this previous ordering.

- The fact that any set H constructed by the **for** loop is a Gröbner Basis for J with respect to the monomial ordering defined by C is proved in both [1] and [18] (where you will also find proofs for the assertions made in the previous paragraph).

- The section of code where we determine the value of t is where we determine the next step of the walk. We choose t to be the minimum value of s in the interval $(0, 1]$ such that, for some polynomial $h \in H$, the ω-degrees of $\text{LT}(h)$ and some other term in h differ, but the $\omega(s)$-degrees of the same two terms are identical. We say that this is the first point on the line segment between the two weight vectors ω and τ where the initial of one of the polynomials in H *degenerates*.

- The success of the Gröbner Walk comes from the fact that it breaks down a Gröbner Basis computation into a series of smaller pieces, each of which computes a Gröbner Basis for a set of initials, a task that is usually quite simple. There are still cases however where this task is complicated and time-consuming, and this has led to the development of so-called *path perturbation* techniques that choose 'easier' paths on which to walk (see for example [1] and [53]).

6.1.3 A Worked Example

Example 6.1.8 Let $F := \{xy - z,\ yz + 2x + z\}$ be a basis generating an ideal J over the polynomial ring $\mathbb{Q}[x, y, z]$. Consider that we want to obtain the Lex Gröbner Basis $H := \{2x + yz + z,\ y^2z + yz + 2z\}$ for J from the DegLex Gröbner Basis $G := \{xy - z,\ yz + 2x + z,\ 2x^2 + xz + z^2\}$ for J using the Gröbner Walk. Utilising Algorithm 17 to do this, we initialise the variables as follows.

[1] The ideal $\text{in}_\omega(J)$ is defined as follows: $p \in J$ if and only if $\text{in}_\omega(p) \in \text{in}_\omega(J)$.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \omega = (1,1,1); \tau = (1,0,0); C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$
$$t = 0; \text{found} = \text{false}.$$

Let us now describe what happens during each pass of the **repeat**... **until** loop of Algorithm 17, noting that as A is equivalent to C to begin with, nothing substantial will happen during the first pass through the loop.

Pass 1

- Construct the set of initials: $G' := \{g'_1,\ g'_2,\ g'_3\} = \{xy,\ yz,\ 2x^2 + xz + z^2\}$ (these are the terms in G that have maximal $(1,1,1)$-degree).
- Compute the Gröbner Basis H' of G' with respect to C.

$$\begin{aligned}
\text{S-pol}(g'_1, g'_2) &= \frac{xyz}{xy}(xy) - \frac{xyz}{yz}(yz) \\
&= 0; \\
\text{S-pol}(g'_1, g'_3) &= \frac{x^2y}{xy}(xy) - \frac{x^2y}{2x^2}(2x^2 + xz + z^2) \\
&= -\frac{1}{2}xyz - \frac{1}{2}yz^2 \\
&\to_{g'_1} -\frac{1}{2}yz^2 \\
&\to_{g'_2} 0; \\
\text{S-pol}(g'_2, g'_3) &= 0 \text{ (by Buchberger's First Criterion).}
\end{aligned}$$

  It follows that $H' = G'$.

- As $H' = G'$, H will also be equal to G, so that $H := \{h_1,\ h_2,\ h_3\} = \{xy - z,\ yz + 2x + z,\ 2x^2 + xz + z^2\}$.
- Let

$$\begin{aligned}
\omega(s) &:= \omega + s(\tau - \omega) \\
&= (1,1,1) + s((1,0,0) - (1,1,1)) \\
&= (1,1,1) + s(0,-1,-1) \\
&= (1, 1 - s, 1 - s).
\end{aligned}$$

To find the next value of t, we must find the minimum value of s such that the $\omega(s)$-degrees of the leading term of a polynomial in H and some other term in the same polynomial agree where their ω-degrees currently differ.

The ω-degrees of the two terms in h_1 differ, so we can seek a value of s such that

$$\begin{aligned} \deg_{\omega(s)}(xy) &= \deg_{\omega(s)}(z) \\ 1 + (1-s) &= (1-s) \\ 1 &= 0 \text{ (inconsistent)}. \end{aligned}$$

For h_2, we have two choices: either

$$\begin{aligned} \deg_{\omega(s)}(yz) &= \deg_{\omega(s)}(x) \\ (1-s) + (1-s) &= 1 \\ 2 - 2s &= 1 \\ s &= \frac{1}{2}; \end{aligned}$$

or

$$\begin{aligned} \deg_{\omega(s)}(yz) &= \deg_{\omega(s)}(z) \\ (1-s) + (1-s) &= (1-s) \\ (1-s) &= 0 \\ s &= 1. \end{aligned}$$

The ω-degrees of all the terms in h_3 are the same, so we can ignore it.

It follows that the minimum value of s (and hence the new value of t) is $\frac{1}{2}$. As this value appears in the interval $(0, 1]$, we set $G = H$; set the new value of ω to be $(1 - \frac{1}{2})(1, 1, 1) + \frac{1}{2}(1, 0, 0) = (1, \frac{1}{2}, \frac{1}{2})$ (and hence change C to be the matrix $\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$); and embark upon a second pass of the **repeat**. . . **until** loop.

Pass 2

- Construct the set of initials: $G' := \{g'_1,\ g'_2,\ g'_3\} = \{xy,\ 2x + yz,\ 2x^2\}$ (these are the

terms in G that have maximal $(1, \frac{1}{2}, \frac{1}{2})$-degree).

- Compute the Gröbner Basis of G' with respect to C.

$$\begin{aligned}
\text{S-pol}(g_1', g_2') &= \frac{xy}{xy}(xy) - \frac{xy}{2x}(2x + yz) \\
&= -\frac{1}{2}y^2z =: g_4'; \\
\text{S-pol}(g_1', g_3') &= \frac{x^2y}{xy}(xy) - \frac{x^2y}{2x^2}(2x^2) \\
&= 0; \\
\text{S-pol}(g_2', g_3') &= \frac{x^2}{2x}(2x + yz) - \frac{x^2}{2x^2}(2x^2) \\
&= \frac{1}{2}xyz \\
&\to_{g_1'} 0; \\
\text{S-pol}(g_1', g_4') &= \frac{xy^2z}{xy}(xy) - \frac{xy^2z}{-\frac{1}{2}y^2z}\left(-\frac{1}{2}y^2z\right) \\
&= 0; \\
\text{S-pol}(g_2', g_4') &= 0 \text{ (by Buchberger's First Criterion)}; \\
\text{S-pol}(g_3', g_4') &= 0 \text{ (by Buchberger's First Criterion)}.
\end{aligned}$$

It follows that $G' = \{g_1',\ g_2',\ g_3',\ g_4'\} = \{xy,\ 2x+yz,\ 2x^2,\ -\frac{1}{2}y^2z\}$ is a Gröbner Basis for $\text{in}_\omega(J)$ with respect to C.

Applying Algorithm 4 to G', we can remove g_1' and g_3' from G' (because $\text{LM}(g_1') = y \times \text{LM}(g_2')$ and $\text{LM}(g_3') = x \times \text{LM}(g_2')$); we must also multiply g_2' and g_4' by $\frac{1}{2}$ and -2 respectively to obtain unit lead coefficients. This leaves us with the unique reduced Gröbner Basis $H' := \{h_1',\ h_2'\} = \{x + \frac{1}{2}yz,\ y^2z\}$ for $\text{in}_\omega(J)$ with respect to C.

- We must now express the two elements of H' in terms of members of G'.

$$\begin{aligned}
h_1' = x + \frac{1}{2}yz &= \frac{1}{2}g_2'; \\
h_2' = y^2z &= -2\left((xy) - \frac{1}{2}y(2x + yz)\right) \text{ (from the S-polynomial)} \\
&= -2\left(g_1' - \frac{1}{2}yg_2'\right).
\end{aligned}$$

Lifting to the full polynomials, h_1' lifts to give the polynomial $h_1 := x + \frac{1}{2}yz + \frac{1}{2}z$; h_2' lifts to give the polynomial $h_2 := -2((xy - z) - \frac{1}{2}y(2x + yz + z)) = -2xy +$

$2z + 2xy + y^2z + yz = y^2z + yz + 2z$; and we are left with the Gröbner Basis $H := \{h_1,\ h_2\} = \{x + \frac{1}{2}yz + \frac{1}{2}z,\ y^2z + yz + 2z\}$ for J with respect to C.

- Let

$$\begin{aligned}
\omega(s) &:= \omega + s(\tau - \omega) \\
&= \left(1, \frac{1}{2}, \frac{1}{2}\right) + s\left((1,0,0) - \left(1, \frac{1}{2}, \frac{1}{2}\right)\right) \\
&= \left(1, \frac{1}{2}, \frac{1}{2}\right) + s\left(0, -\frac{1}{2}, -\frac{1}{2}\right) \\
&= \left(1, \frac{1}{2}(1-s), \frac{1}{2}(1-s)\right).
\end{aligned}$$

Finding the minimum value of s, for h_1 we can have

$$\begin{aligned}
\deg_{\omega(s)}(x) &= \deg_{\omega(s)}(z) \\
1 &= \frac{1}{2}(1-s) \\
s &= -1 \text{ (undefined: we must have } s \in (0,1]).
\end{aligned}$$

Continuing with h_2, we can either have

$$\begin{aligned}
\deg_{\omega(s)}(y^2z) &= \deg_{\omega(s)}(yz) \\
3\left(\frac{1}{2}(1-s)\right) &= 2\left(\frac{1}{2}(1-s)\right) \\
\frac{1}{2}(1-s) &= 0 \\
s &= 1;
\end{aligned}$$

or

$$\begin{aligned}
\deg_{\omega(s)}(y^2z) &= \deg_{\omega(s)}(z) \\
3\left(\frac{1}{2}(1-s)\right) &= \frac{1}{2}(1-s) \\
1-s &= 0 \\
s &= 1.
\end{aligned}$$

It follows that the minimum value of s (and hence the new value of t) is 1. As this value appears in the interval $(0,1]$, we set $G = H$; set the new value of ω

to be $(1-1)(1,\frac{1}{2},\frac{1}{2})+1(1,0,0)=(1,0,0)$ (and hence change C to be the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$); and embark upon a third (and final) pass of the **repeat**. . . **until** loop.

Pass 3

- Construct the set of initials: $G' := \{g_1',\, g_2'\} = \{x,\ y^2z+yz+2z\}$ (these are the terms in G that have maximal $(1,0,0)$-degree).

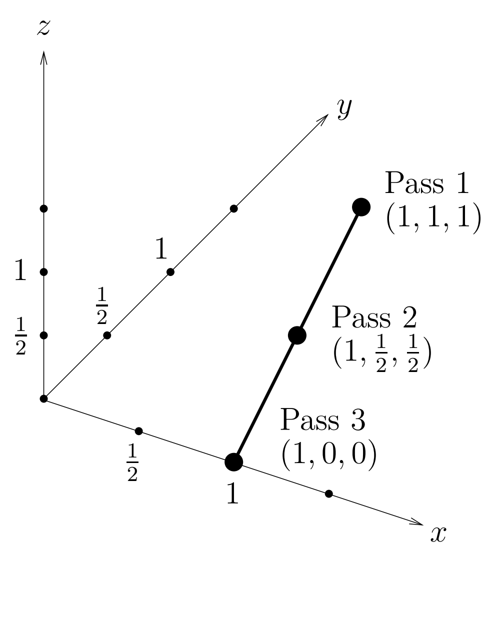
- Compute the Gröbner Basis H' of G' with respect to C.

  $$\text{S-pol}(g_1', g_2') = 0 \text{ (by Buchberger's First Criterion).}$$

  It follows that $H' = G'$.

- As $H' = G'$, H will also be equal to G, so that $H := \{h_1,\, h_2\} = \{x+\frac{1}{2}yz+\frac{1}{2}z,\ y^2z+yz+2z\}$. Further, as t is now equal to 1, we have arrived at our target ordering (Lex) and can return H as the output Gröbner Basis, a basis that is equivalent to the Lex Gröbner Basis given for J at the beginning of this example.

We can summarise the path taken during the walk in the following diagram.



Algorithm 18 The Commutative Involutive Walk Algorithm

Input: An Involutive Basis $G = \{g_1, g_2, \ldots, g_m\}$ with respect to an involutive division I and an admissible monomial ordering O with an associated matrix A, where G generates an ideal J over a commutative polynomial ring $\mathcal{R} = R[x_1, \ldots, x_n]$.

Output: An Involutive Basis $H = \{h_1, h_2, \ldots, h_p\}$ for J with respect to I and an admissible monomial ordering O' with an associated matrix B.

Let ω and τ be the weight vectors corresponding to the first rows of A and B;
Let C be the matrix whose first row is equal to ω and whose remainder is equal to the whole of the matrix B;
$t = 0$; found = false;
repeat
  Let $G' = \{\mathrm{in}_\omega(g_i)\}$ for all $g_i \in G$;
  Compute an Involutive Basis H' for G' with respect to the monomial ordering defined by the matrix C (use Algorithm 9);
  $H = \emptyset$;
  for each $h' \in H'$ **do**
    Let $\sum_{i=1}^{j} p_i g'_i$ be the logged representation of h' with respect to G' (where $g'_i \in G'$ and $p_i \in \mathcal{R}$), obtained either through computing a Logged Involutive Basis above or by involutively dividing h' with respect to G';
    $H = H \cup \{\sum_{i=1}^{j} p_i g_i\}$, where $\mathrm{in}_\omega(g_i) = g'_i$;
  end for
  if $(t == 1)$ **then**
    found = true;
  else
    $t = \min(\{s \mid \deg_{\omega(s)}(p_1) = \deg_{\omega(s)}(p_i), \deg_{\omega(0)}(p_1) \neq \deg_{\omega(0)}(p_i),$
    $h = p_1 + \cdots + p_k \in H\} \cap (0, 1])$, where $\omega(s) := \omega + s(\tau - \omega)$ for $0 \leqslant s \leqslant 1$;
  end if
  if (t is undefined) **then**
    found = true;
  else
    $G = H$; $\omega = (1 - t)\omega + t\tau$;
  end if
until (found = true)
return H;

6.1.4 The Commutative Involutive Walk Algorithm

In [30], Golubitsky generalised the Gröbner Walk technique to give a method for converting an Involutive Basis with respect to one monomial ordering to an Involutive Basis with respect to another monomial ordering. Algorithmically, the way in which we perform this *Involutive Walk* is virtually identical to the way we perform the Gröbner Walk, as can be seen by comparing Algorithms 17 and 18. The change however comes when proving the correctness of the algorithm, as we have to show that each G' is an Involutive Basis for $\mathrm{in}_\omega(J)$ and that each H is an Involutive Basis for J (see [30] for these proofs).

6.2 Noncommutative Walks

In the commutative case, any monomial ordering can be represented by a matrix that provides a decomposition of the ordering in terms of the rows of the matrix. This decomposition is then utilised in the Gröbner Walk algorithm when (for example) we use the first row of the matrix to provide a set of initials for a particular basis G (cf. Definition 6.1.6).

In the noncommutative case, because monomials cannot be represented by multidegrees, monomial orderings cannot be represented by matrices. This seems to shut the door on any generalisation of the Gröbner Walk to the noncommutative case, as not only is there no first row of a matrix to provide a set of initials, but no notion of a walk between two matrices can be formulated either.

Despite this, we note that in the commutative case, if the first rows of the source and target matrices are the same, then the Gröbner Walk will complete in one pass of the algorithm, and all that is needed is the first row of the source matrix to provide a set of initials to work with.

Generalising to the noncommutative case, it is possible that if we can find a way to decompose a noncommutative monomial ordering to provide a set of initials to work with, then a noncommutative Gröbner Walk algorithm could complete in one pass if the source and target monomial orderings used the same method to compute sets of initials.

6.2.1 Functional Decompositions

Considering the monomial orderings defined in Section 1.2.2, we note that all the orderings are defined step-by-step. For example, the DegLex monomial ordering compares two monomials by degree first, then by the first letter of each monomial, then by the second letter, and so on. This provides us with an opportunity to decompose each monomial ordering into a series of steps or functions, a decomposition we shall term a *functional decomposition*.

Definition 6.2.1 An *ordering function* is a function

$$\theta : M \longrightarrow \mathbb{Z}$$

that assigns an integer to any monomial $m \in M$, where M denotes the set of all monomials over a polynomial ring $R\langle x_1, \ldots, x_n\rangle$. We call the integer assigned by θ to m the θ-value of m.

Remark 6.2.2 The θ-value of any term will be equal to the θ-value of the term's associated monomial.

Definition 6.2.3 A *functional decomposition* Θ is a (possibly infinite) sequence of ordering functions, written $\Theta = \{\theta_1, \theta_2, \ldots\}$.

Definition 6.2.4 Let O be a monomial ordering; let m_1 and m_2 be two arbitrary monomials such that $m_1 < m_2$ with respect to O; and let $\Theta = \{\theta_1, \theta_2, \ldots\}$ be a functional decomposition. We say that Θ defines O if and only if $\theta_i(m_1) < \theta_i(m_2)$ for some $i \geqslant 1$ and $\theta_j(m_1) = \theta_j(m_2)$ for all $1 \leqslant j < i$.

To describe the functional decompositions corresponding to the monomial orderings defined in Section 1.2.2, we first need the following definition.

Definition 6.2.5 Let m be an arbitrary monomial over a polynomial ring $R\langle x_1, \ldots, x_n\rangle$. The i-th valuing function of m, written $\mathrm{val}_i(m)$, is an ordering function that assigns an integer to m as follows.

$$\mathrm{val}_i(m) = \begin{cases} j & \text{if } \mathrm{Subword}(m, i, i) = x_j \text{ (where } 1 \leqslant j \leqslant n). \\ n+1 & \text{if } \mathrm{Subword}(m, i, i) \text{ is undefined.} \end{cases}$$

Let us now describe the functional decompositions corresponding to those monomial orderings defined in Section 1.2.2 that are admissible.

Definition 6.2.6 The functional decomposition $\Theta = \{\theta_1, \theta_2, \ldots\}$ corresponding to the DegLex monomial ordering is defined (for an arbitrary monomial m) as follows.

$$\theta_i(m) = \begin{cases} \deg(m) & \text{if } i = 1. \\ n + 1 - \mathrm{val}_{i-1}(m) & \text{if } i > 1. \end{cases}$$

Similarly, we can define DegInvLex by

$$\theta_i(m) = \begin{cases} \deg(m) & \text{if } i = 1. \\ \mathrm{val}_{i-1}(m) & \text{if } i > 1. \end{cases}$$

and DegRevLex by

$$\theta_i(m) = \begin{cases} \deg(m) & \text{if } i = 1. \\ \mathrm{val}_{\deg(m)+2-i}(m) & \text{if } i > 1. \end{cases}$$

Example 6.2.7 Let $m_1 := xyxz^2$ and $m_2 := xzyz^2$ be two monomials over the polynomial ring $\mathbb{Q}\langle x, y, z\rangle$. With respect to DegLex, we can work out that $xyxz^2 > xzyz^2$, because $\theta_1(m_1) = \theta_1(m_2)$ (or $\deg(m_1) = \deg(m_2)$); $\theta_2(m_1) = \theta_2(m_2)$ (or $n+1-\mathrm{val}_1(m_1) = n+1-\mathrm{val}_1(m_2)$, $3+1-1 = 3+1-1$); and $\theta_3(m_1) > \theta_3(m_2)$ (or $n+1-\mathrm{val}_2(m_1) > n+1-\mathrm{val}_2(m_2)$, $3+1-2 > 3+1-3$). Similarly, with respect to DegInvLex, we can work out that $xyxz^2 < xzyz^2$ (because $\theta_3(m_1) < \theta_3(m_2)$, or $2 < 3$); and with respect to DegRevLex, we can work out that $xyxz^2 < xzyz^2$ (because $\theta_4(m_1) < \theta_4(m_2)$, or $1 < 2$).

Definition 6.2.8 Given a polynomial p and an ordering function θ, the *initial* of p with respect to θ, written $\mathrm{in}_\theta(p)$, is made up of those terms in p that have maximal θ-value. For example, if θ is the degree function and if $p = x^4 + zxy^2 + y^3 + z^2x$, then $\mathrm{in}_\theta(p) = x^4 + zxy^2$.

Definition 6.2.9 Given an ordering function θ, a polynomial p is said to be θ-*homogeneous* if $p = \mathrm{in}_\theta(p)$.

Definition 6.2.10 An ordering function θ is *compatible* with a monomial ordering O if, given any polynomial $p = t_1 + \cdots + t_m$ ordered in descending order with respect to O, $\theta(t_1) \geqslant \theta(t_i)$ holds for all $1 < i \leqslant m$.

Definition 6.2.11 An ordering function θ is *extendible* if, given any θ-homogeneous polynomial p, any multiple upv of p by terms u and v is also θ-homogeneous.

Remark 6.2.12 Of the ordering functions encountered so far, only the degree function, val_1 and[2] $\mathrm{val}_{\deg(m)}$ (for any given monomial m) are extendible.

Definition 6.2.13 Two noncommutative monomial orderings O_1 and O_2 are said to be *harmonious* if (i) there exist functional decompositions $\Theta_1 = \{\theta_{1_1}, \theta_{1_2}, \ldots\}$ and $\Theta_2 = \{\theta_{2_1}, \theta_{2_2}, \ldots\}$ defining O_1 and O_2 respectively; and (ii) the ordering functions θ_{1_1} and θ_{2_1} are identical and extendible.

Remark 6.2.14 The noncommutative monomial orderings DegLex, DegInvLex and DegRevLex are all (pairwise) harmonious.

6.2.2 The Noncommutative Gröbner Walk Algorithm for Harmonious Monomial Orderings

We present in Algorithm 19 an algorithm to perform a Gröbner Walk between two harmonious noncommutative monomial orderings.

Termination of Algorithm 19 depends on the termination of Algorithm 5 as used (in Algorithm 19) to compute a noncommutative Gröbner Basis for the set G'. The correctness of Algorithm 19 is provided by the following two propositions.

Proposition 6.2.15 *G' is always a Gröbner Basis for the ideal*[3] $\mathrm{in}_\theta(J)$ *with respect to the monomial ordering O.*

Proof: Because θ is compatible with O (by definition), the S-polynomials involving members of G will be in one-to-one correspondence with the S-polynomials involving members of G', with the same monomial being 'cancelled' in each pair of corresponding S-polynomials.

Let p be an arbitrary S-polynomial involving members of G (with corresponding S-polynomial q involving members of G'). Because G is a Gröbner Basis for J with respect

[2] Think of $\mathrm{val}_{\deg(m)}$ as finding the value of the final variable in m (as opposed to val_1 finding the value of the first variable in m).

[3] The ideal $\mathrm{in}_\theta(J)$ is defined as follows: $p \in J$ if and only if $\mathrm{in}_\theta(p) \in \mathrm{in}_\theta(J)$.

Algorithm 19 The Noncommutative Gröbner Walk Algorithm for Harmonious Monomial Orderings

Input: A Gröbner Basis $G = \{g_1, g_2, \ldots, g_m\}$ with respect to an admissible monomial ordering O with an associated functional decomposition A, where G generates an ideal J over a noncommutative polynomial ring $\mathcal{R} = R\langle x_1, \ldots, x_n\rangle$.

Output: A Gröbner Basis $H = \{h_1, h_2, \ldots, h_p\}$ for J with respect to an admissible monomial ordering O' with an associated functional decomposition B, where O and O' are harmonious.

Let θ be the ordering function corresponding to the first ordering function of A;

Let $G' = \{\mathrm{in}_\theta(g_i)\}$ for all $g_i \in G$;

Compute a reduced Gröbner Basis H' for G' with respect to the monomial ordering O' (use Algorithms 5 and 6);

$H = \emptyset$;

for each $h' \in H'$ **do**

Let $\sum_{i=1}^{j} \ell_i g_i' r_i$ be the logged representation of h' with respect to G' (where $g_i' \in G'$ and the ℓ_i and the r_i are terms), obtained either through computing a Logged Gröbner Basis above or by dividing h' with respect to G';

$H = H \cup \{\sum_{i=1}^{j} \ell_i g_i r_i\}$, where $\mathrm{in}_\theta(g_i) = g_i'$;

end for

Reduce H with respect to O' (use Algorithm 6);

return H;

to O, p will reduce to zero using G by the series of reductions

$$p \rightarrow_{g_{i_1}} p_1 \rightarrow_{g_{i_2}} p_2 \rightarrow_{g_{i_3}} \cdots \rightarrow_{g_{i_\alpha}} 0,$$

where $g_{i_j} \in G$ for all $1 \leqslant j \leqslant \alpha$.

Claim: q will reduce to zero using G' (and hence G' is a Gröbner Basis for $\mathrm{in}_\theta(J)$ with respect to O by Definition 3.1.8) by the series of reductions

$$q \rightarrow_{\mathrm{in}_\theta(g_{i_1})} q_1 \rightarrow_{\mathrm{in}_\theta(g_{i_2})} q_2 \rightarrow_{\mathrm{in}_\theta(g_{i_3})} \cdots \rightarrow_{\mathrm{in}_\theta(g_{i_\beta})} 0,$$

where $0 \leqslant \beta \leqslant \alpha$.

Proof of Claim: Let w be the overlap word associated to the S-polynomial p. If $\theta(\mathrm{LM}(p)) < \theta(w)$, then because θ is extendible it is clear that $q = 0$, and so the proof is complete. Otherwise, we must have $q = \mathrm{in}_\theta(p)$, and so by the compatibility of θ with O, we can use the polynomial $\mathrm{in}_\theta(g_{i_1}) \in G'$ to reduce q to give the polynomial q_1. We now proceed by induction (if $\theta(\mathrm{LM}(p_1)) < \theta(\mathrm{LM}(p))$ then $q_1 = 0, \dots$), noting that the process will terminate because $\mathrm{in}_\theta(p_\alpha = 0) = 0$. □

Proposition 6.2.16 *The set H constructed by the* **for** *loop of Algorithm 19 is a Gröbner Basis for J with respect to the monomial ordering O'.*

Proof: By Definition 3.1.8, we can show that H is a Gröbner Basis for J by showing that all S-polynomials involving members of H reduce to zero using H. Assume for a contradiction that an S-polynomial p involving members of H does not reduce to zero using H, and instead only reduces to a polynomial $q \neq 0$.

As all members of H are members of the ideal J (by the way H was constructed as combinations of elements of G), it is clear that q is also a member of the ideal J, as all we have done in constructing q is to reduce a combination of two members of H with respect to H. It follows that the polynomial $\mathrm{in}_\theta(q)$ is a member of the ideal $\mathrm{in}_\theta(J)$.

Because H' is a Gröbner Basis for the ideal $\mathrm{in}_\theta(J)$ with respect to O', there must be a polynomial $h' \in H'$ such that $h' \mid \mathrm{in}_\theta(q)$. Let $\sum_{i=1}^{j} \ell_i g'_i r_i$ be the logged representation of

h' with respect to G'. Then it is clear that

$$\sum_{i=1}^{j} \ell_i g_i' r_i \mid \mathrm{in}_\theta(q).$$

However θ is compatible with O', so that

$$\sum_{i=1}^{j} \ell_i g_i r_i \mid q.$$

It follows that there exists a polynomial $h \in H$ dividing our polynomial q, contradicting our initial assumption. $\square$

6.2.3 A Worked Example

Example 6.2.17 Let $F := \{x^2 + y^2 + 8,\ 2xy + y^2 + 5\}$ be a basis generating an ideal J over the polynomial ring $\mathbb{Q}\langle x, y\rangle$. Consider that we want to obtain the DegLex Gröbner Basis $H := \{2xy + y^2 + 5,\ x^2 + y^2 + 8,\ 5y^3 - 10x + 37y,\ 2yx + y^2 + 5\}$ for J from the DegRevLex Gröbner Basis $G := \{2xy - x^2 - 3,\ y^2 + x^2 + 8,\ 5x^3 + 6y + 35x,\ 2yx - x^2 - 3\}$ for J using the Gröbner Walk. Utilising Algorithm 19 to do this, we initialise θ to be the degree function and we proceed as follows.

- Construct the set of initials: $G' := \{g_1',\ g_2',\ g_3',\ g_4'\} = \{-x^2+2xy,\ x^2+y^2,\ 5x^3,\ -x^2+2yx\}$ (these are the terms in G that have maximal degree).
- Compute the Gröbner Basis of G' with respect to the DegLex monomial ordering (for simplicity, we will not provide details of those S-polynomials that reduce to zero

or can be ignored due to Buchberger's Second Criterion).

$$
\begin{aligned}
\text{S-pol}(1, g'_1, 1, g'_2) &= (-x^2 + 2xy) - (-1)(x^2 + y^2) \\
&= 2xy + y^2 =: g'_5; \\
\text{S-pol}(1, g'_1, 1, g'_4) &= (-1)(-x^2 + 2xy) - (-1)(-x^2 + 2yx) \\
&= -2xy + 2yx \\
&\to_{g'_5} -2xy + 2yx + (2xy + y^2) \\
&= 2yx + y^2 =: g'_6; \\
\text{S-pol}(y, g'_1, 1, g'_6) &= 2y(-x^2 + 2xy) - (-1)(2yx + y^2)x \\
&= 4yxy + y^2x \\
&\to_{g'_5} 4yxy + y^2x - 2y(2xy + y^2) \\
&= y^2x - 2y^3 \\
&\to_{g'_6} y^2x - 2y^3 - \frac{1}{2}y(2yx + y^2) \\
&= -\frac{5}{2}y^3 =: g'_7.
\end{aligned}
$$

After g'_7 is added to the current basis, all S-polynomials now reduce to zero, leaving the Gröbner Basis $G' = \{g'_1,\ g'_2,\ g'_3,\ g'_4,\ g'_5,\ g'_6,\ g'_7\} = \{-x^2+2xy,\ x^2+y^2,\ 5x^3,\ -x^2+2yx,\ 2xy + y^2,\ 2yx + y^2,\ -\frac{5}{2}y^3\}$ for $\text{in}_\theta(J)$ with respect to O'.

Applying Algorithm 6 to G', we can remove g'_1, g'_2 and g'_3 from G' (because their lead monomials are all multiples of $\text{LM}(g'_4)$); we must multiply g'_4, g'_5, g'_6 and g'_7 by -1, $\frac{1}{2}$, $\frac{1}{2}$ and $-\frac{2}{5}$ respectively (to obtain unit lead coefficients); and the polynomial g'_4 can (then) be further reduced as follows.

$$
\begin{aligned}
g'_4 &= x^2 - 2yx \\
&\to_{g'_6} x^2 - 2yx + 2\left(yx + \frac{1}{2}y^2\right) \\
&= x^2 + y^2.
\end{aligned}
$$

This leaves us with the unique reduced Gröbner Basis $H' := \{h'_1,\ h'_2,\ h'_3,\ h'_4\} = \{x^2 + y^2,\ xy + \frac{1}{2}y^2,\ yx + \frac{1}{2}y^2,\ y^3\}$ for $\text{in}_\theta(J)$ with respect to O'.

- We must now express the four elements of H' in terms of members of G'.

$$\begin{aligned}
h_1' = x^2 + y^2 &= g_2';\\
h_2' = xy + \frac{1}{2}y^2 &= \frac{1}{2}(g_1' + g_2') \text{ (from the S-polynomial);}\\
h_3' = yx + \frac{1}{2}y^2 &= \frac{1}{2}(-g_1' + g_4' + (g_1' + g_2'))\\
&= \frac{1}{2}(g_2' + g_4');\\
h_4' = y^3 &= -\frac{2}{5}\left(2y(g_1') + (g_2' + g_4')x - 2y(g_1' + g_2') - \frac{1}{2}y(g_2' + g_4')\right)\\
&= -\frac{2}{5}\left(g_2'x - \frac{5}{2}yg_2' + g_4'x - \frac{1}{2}yg_4'\right).
\end{aligned}$$

Lifting to the full polynomials, we obtain the Gröbner Basis $H := \{h_1, h_2, h_3, h_4\}$ as follows.

$$\begin{aligned}
h_1 &= g_2\\
&= x^2 + y^2 + 8;\\
h_2 &= \frac{1}{2}(g_1 + g_2)\\
&= \frac{1}{2}(-x^2 + 2xy - 3 + x^2 + y^2 + 8)\\
&= xy + \frac{1}{2}y^2 + \frac{5}{2};\\
h_3 &= \frac{1}{2}(g_2 + g_4)\\
&= \frac{1}{2}(x^2 + y^2 + 8 - x^2 + 2yx - 3)\\
&= yx + \frac{1}{2}y^2 + \frac{5}{2};\\
h_4 &= -\frac{2}{5}\left(g_2x - \frac{5}{2}yg_2 + g_4x - \frac{1}{2}yg_4\right)\\
&= -\frac{2}{5}\left(x^3 + y^2x + 8x - \frac{5}{2}yx^2 - \frac{5}{2}y^3 - 20y\right.\\
&\qquad \left. -x^3 + 2yx^2 - 3x + \frac{1}{2}yx^2 - y^2x + \frac{3}{2}y\right)\\
&= y^3 - 2x + \frac{37}{5}y.
\end{aligned}$$

The set H does not reduce any further, so we return the output DegLex Gröbner Basis $\{x^2 + y^2 + 8,\ xy + \frac{1}{2}y^2 + \frac{5}{2},\ yx + \frac{1}{2}y^2 + \frac{5}{2},\ y^3 - 2x + \frac{37}{5}y\}$ for J, a basis

that is equivalent to the DegLex Gröbner Basis given for J at the beginning of this example.

6.2.4 The Noncommutative Involutive Walk Algorithm for Harmonious Monomial Orderings

We present in Algorithm 20 an algorithm to perform an Involutive Walk between two harmonious noncommutative monomial orderings.

Algorithm 20 The Noncommutative Involutive Walk Algorithm for Harmonious Monomial Orderings

Input: An Involutive Basis $G = \{g_1, g_2, \ldots, g_m\}$ with respect to an involutive division I and an admissible monomial ordering O with an associated functional decomposition A, where G generates an ideal J over a noncommutative polynomial ring $\mathcal{R} = R\langle x_1, \ldots, x_n\rangle$.

Output: An Involutive Basis $H = \{h_1, h_2, \ldots, h_p\}$ for J with respect to I and an admissible monomial ordering O' with an associated functional decomposition B, where O and O' are harmonious.

Let θ be the ordering function corresponding to the first ordering function of A;

Let $G' = \{\mathrm{in}_\theta(g_i)\}$ for all $g_i \in G$;

Compute an Involutive Basis H' for G' with respect to I and the monomial ordering O' (use Algorithm 12);

$H = \emptyset$;

for each $h' \in H'$ **do**

  Let $\sum_{i=1}^{j} \ell_i g_i' r_i$ be the logged representation of h' with respect to G' (where $g_i' \in G'$ and the ℓ_i and the r_i are terms), obtained either through computing a Logged Involutive Basis above or by involutively dividing h' with respect to G';

  $H = H \cup \{\sum_{i=1}^{j} \ell_i g_i r_i\}$, where $\mathrm{in}_\theta(g_i) = g_i'$;

end for

return H;

Termination of Algorithm 20 depends on the termination of Algorithm 12 as used (in Algorithm 20) to compute a noncommutative Involutive Basis for the set G'. The correctness of Algorithm 20 is provided by the following two propositions.

Proposition 6.2.18 *G' is always an Involutive Basis for the ideal $\mathrm{in}_\theta(J)$ with respect to I and the monomial ordering O.*

Proof: Let $p := ugv$ be an arbitrary multiple of a polynomial $g \in G$ by terms u and v. Because G is an Involutive Basis for J with respect to I and O, p will involutively reduce to zero using G by the series of involutive reductions

$$p \xrightarrow[I\ \ g_{i_1}]{} p_1 \xrightarrow[I\ \ g_{i_2}]{} p_2 \xrightarrow[I\ \ g_{i_3}]{} \cdots \xrightarrow[I\ \ g_{i_\alpha}]{} 0,$$

where $g_{i_j} \in G$ for all $1 \leqslant j \leqslant \alpha$.

Claim: The polynomial $q := u\mathrm{in}_\theta(g)v$ will involutively reduce to zero using G' (and hence G' is an Involutive Basis for $\mathrm{in}_\theta(J)$ with respect to I and O by Definition 5.2.7) by the series of involutive reductions

$$q \xrightarrow[I\ \ \mathrm{in}_\theta(g_{i_1})]{} q_1 \xrightarrow[I\ \ \mathrm{in}_\theta(g_{i_2})]{} q_2 \xrightarrow[I\ \ \mathrm{in}_\theta(g_{i_3})]{} \cdots \xrightarrow[I\ \ \mathrm{in}_\theta(g_{i_\beta})]{} 0,$$

where $1 \leqslant \beta \leqslant \alpha$.

Proof of Claim: Because θ is extendible, it is clear that $q = \mathrm{in}_\theta(p)$. Further, because θ is compatible with O (by definition), the multiplicative variables of G and G' with respect to I will be identical, and so it follows that because the polynomial $g_{i_1} \in G$ was used to involutively reduce p to give the polynomial p_1, then the polynomial $\mathrm{in}_\theta(g_{i_1}) \in G'$ can be used to involutively reduce q to give the polynomial q_1.

If $\theta(\mathrm{LM}(p_1)) < \theta(\mathrm{LM}(p))$, then because θ is extendible it is clear that $q_1 = 0$, and so the proof is complete. Otherwise, we must have $q_1 = \mathrm{in}_\theta(p_1)$, and so (again) by the compatibility of θ with O, we can use the polynomial $\mathrm{in}_\theta(g_{i_2}) \in G'$ to involutively reduce q_1 to give the polynomial q_2. We now proceed by induction (if $\theta(\mathrm{LM}(p_2)) < \theta(\mathrm{LM}(p_1))$ then $q_2 = 0, \ldots$), noting that the process will terminate because $\mathrm{in}_\theta(p_\alpha = 0) = 0$. □

Proposition 6.2.19 *The set H constructed by the* **for** *loop of Algorithm 20 is an Involutive Basis for J with respect to I and the monomial ordering O'.*

Proof: By Definition 5.2.7, we can show that H is an Involutive Basis for J by showing that any multiple upv of any polynomial $p \in H$ by any terms u and v involutively reduces to zero using H. Assume for a contradiction that such a multiple does not involutively reduce to zero using H, and instead only involutively reduces to a polynomial $q \neq 0$.

As all members of H are members of the ideal J (by the way H was constructed as combinations of elements of G), it is clear that q is also a member of the ideal J, as all we have done in constructing q is to reduce a multiple of a polynomial in H with respect to H. It follows that the polynomial $\text{in}_\theta(q)$ is a member of the ideal $\text{in}_\theta(J)$.

Because H' is an Involutive Basis for the ideal $\text{in}_\theta(J)$ with respect to I and O', there must be a polynomial $h' \in H'$ such that $h' \mid_I \text{in}_\theta(q)$. Let $\sum_{i=1}^{j} \ell_i g_i' r_i$ be the logged representation of h' with respect to G'. Then it is clear that

$$\sum_{i=1}^{j} \ell_i g_i' r_i \mid_I \text{in}_\theta(q).$$

However θ is compatible with O' (in particular the multiplicative variables for H' and H with respect to I and O' will be identical), so that

$$\sum_{i=1}^{j} \ell_i g_i r_i \mid_I q.$$

It follows that there exists a polynomial $h \in H$ involutively dividing our polynomial q, contradicting our initial assumption. □

6.2.5 A Worked Example

Example 6.2.20 Let $F := \{x^2+y^2+8,\ 2xy+y^2+5\}$ be a basis generating an ideal J over the polynomial ring $\mathbb{Q}\langle x, y\rangle$. Consider that we want to obtain the DegRevLex Involutive Basis $H := \{2xy - x^2 - 3,\ y^2 + x^2 + 8,\ 5x^3 + 6y + 35x,\ 5yx^2 + 3y + 10x,\ 2yx - x^2 - 3\}$ for J from the DegLex Involutive Basis $G := \{2xy + y^2 + 5,\ x^2 + y^2 + 8,\ 5y^3 - 10x + 37y,\ 5xy^2 + 5x - 6y,\ 2yx + y^2 + 5\}$ for J using the Involutive Walk, where H and G are both Involutive Bases with respect to the left division $\lhd$. Utilising Algorithm 20 to do this, we initialise θ to be the degree function and we proceed as follows.

- Construct the set of initials:

$$G' := \{g_1',\ g_2',\ g_3',\ g_4',\ g_5'\} = \{y^2 + 2xy,\ y^2 + x^2,\ 5y^3,\ 5xy^2,\ y^2 + 2yx\}$$

  (these are the terms in G that have maximal degree).

- Compute the Involutive Basis of G' with respect to $\lhd$ and the DegRevLex monomial

ordering. Step 1: autoreduce the set G'.

$$\begin{array}{rcl}
g_1' & = & y^2 + 2xy \\
& \xrightarrow[\lhd\; g_2']{} & y^2 + 2xy - (y^2 + x^2) \\
& = & 2xy - x^2 =: g_6'; \\
G' & = & (G' \setminus \{g_1'\}) \cup \{g_6'\}; \\
g_2' & = & y^2 + x^2 \\
& \xrightarrow[\lhd\; g_5']{} & y^2 + x^2 - (y^2 + 2yx) \\
& = & -2yx + x^2 =: g_7'; \\
G' & = & (G' \setminus \{g_2'\}) \cup \{g_7'\}; \\
g_3' & = & 5y^3 \\
& \xrightarrow[\lhd\; g_5']{} & 5y^3 - 5y(y^2 + 2yx) \\
& = & -10y^2x \\
& \xrightarrow[\lhd\; g_7']{} & -10y^2x - 5y(-2yx + x^2) \\
& = & -5yx^2 =: g_8'; \\
G' & = & (G' \setminus \{g_3'\}) \cup \{g_8'\}; \\
g_4' & = & 5xy^2 \\
& \xrightarrow[\lhd\; g_5']{} & 5xy^2 - 5x(y^2 + 2yx) \\
& = & -10xyx \\
& \xrightarrow[\lhd\; g_7']{} & -10xyx - 5x(-2yx + x^2) \\
& = & -5x^3 =: g_9'; \\
G' & = & (G' \setminus \{g_4'\}) \cup \{g_9'\}; \\
g_5' & = & y^2 + 2yx \\
& \xrightarrow[\lhd\; g_7']{} & y^2 + 2yx + (-2yx + x^2) \\
& = & y^2 + x^2 =: g_{10}'; \\
G' & = & (G' \setminus \{g_5'\}) \cup \{g_{10}'\}.
\end{array}$$

Step 2: process the prolongations of the set $G' = \{g_6',\ g_7',\ g_8',\ g_9',\ g_{10}'\}$. Because all ten of these prolongations involutively reduce to zero using G', we are left with the Involutive Basis $H' := \{h_1',\ h_2',\ h_3',\ h_4',\ h_5'\} = \{2xy - x^2,\ -2yx + x^2,\ -5yx^2,\ -5x^3,\ y^2 +$

$x^2\}$ for $\text{in}_\theta(J)$ with respect to $\lhd$ and O'.

- We must now express the five elements of H' in terms of members of G'.

$$\begin{aligned}
h_1' = 2xy - x^2 &= g_1' - g_2' \text{ (from autoreduction)};\\
h_2' = -2yx + x^2 &= g_2' - g_5';\\
h_3' = -5yx^2 &= g_3' - 5yg_5' - 5y(g_2' - g_5')\\
&= -5yg_2' + g_3';\\
h_4' = -5x^3 &= g_4' - 5xg_5' - 5x(g_2' - g_5')\\
&= -5xg_2' + g_4';\\
h_5' = y^2 + x^2 &= g_5' + (g_2' - g_5')\\
&= g_2'.
\end{aligned}$$

Lifting to the full polynomials, we obtain the Involutive Basis $H := \{h_1,\ h_2,\ h_3,\ h_4,\ h_5\}$ as follows.

$$\begin{aligned}
h_1 &= g_1 - g_2\\
&= (y^2 + 2xy + 5) - (y^2 + x^2 + 8)\\
&= 2xy - x^2 - 3;\\
h_2 &= g_2 - g_5\\
&= (y^2 + x^2 + 8) - (y^2 + 2yx + 5)\\
&= -2yx + x^2 + 3;\\
h_3 &= -5yg_2 + g_3\\
&= -5y(y^2 + x^2 + 8) + (5y^3 + 37y - 10x)\\
&= -5yx^2 - 3y - 10x;\\
h_4 &= -5xg_2 + g_4\\
&= -5x(y^2 + x^2 + 8) + (5xy^2 - 6y + 5x)\\
&= -5x^3 - 6y - 35x;\\
h_5 &= g_2\\
&= y^2 + x^2 + 8.
\end{aligned}$$

We can now return the output DegRevLex Involutive Basis $H = \{2xy - x^2 - 3,\ -2yx + x^2 + 3,\ -5yx^2 - 3y - 10x,\ -5x^3 - 6y - 35x,\ y^2 + x^2 + 8\}$ for J with

respect to $\lhd$, a basis that is equivalent to the DegRevLex Involutive Basis given for J at the beginning of this example.

6.2.6 Noncommutative Walks Between Any Two Monomial Orderings?

Thus far, we have only been able to define a noncommutative walk between two harmonious monomial orderings, where we recall that the first ordering functions of the functional decompositions of the two monomial orderings must be identical and extendible. For walks between two arbitrary monomial orderings, the first ordering functions need not be identical any more, but it is clear that they must still be extendible, so that (in an algorithm to perform such a walk) each basis G' is a Gröbner Basis for each ideal $\text{in}_\theta(J)$ (compare with the proofs of Propositions 6.2.15 and 6.2.18). This condition will also apply to any 'intermediate' monomial ordering we will encounter during the walk, but the challenge will be in how to define these intermediate orderings, so that we generalise the commutative concept of choosing a weight vector ω_{i+1} on the line segment between two weight vectors ω_i and τ.

Open Question 4 Is it possible to perform a noncommutative walk between two admissible and extendible monomial orderings that are not harmonious?

Chapter 7

Conclusions

7.1 Current State of Play

The goal of this thesis was to combine the theories of noncommutative Gröbner Bases and commutative Involutive Bases to give a theory of noncommutative Involutive Bases. To accomplish this, we started by surveying the background theory in Chapters 1 to 4, focusing our account on the various algorithms associated with the theory. In particular, we mentioned several improvements to the standard algorithms, including how to compute commutative Involutive Bases by homogeneous methods, which required the introduction of a new property (extendibility) of commutative involutive divisions.

The theory of noncommutative Involutive Bases was introduced in Chapter 5, where we described how to perform noncommutative involutive reduction (Definition 5.1.1 and Algorithm 10); introduced the notion of a noncommutative involutive division (Definition 5.1.6); described what is meant by a noncommutative Involutive Basis (Definition 5.2.7); and gave an algorithm to compute noncommutative Involutive Bases (Algorithm 12). Several noncommutative involutive divisions were also defined, each of which was shown to satisfy certain properties (such as continuity) allowing the deductions that all Locally Involutive Bases are Involutive Bases; and that all Involutive Bases are Gröbner Bases.

To finish, we partially generalised the theory of the Gröbner Walk to the noncommutative case in Chapter 6, yielding both Gröbner and Involutive Walks between harmonious noncommutative monomial orderings.

7.2 Future Directions

As well as answering a few questions, the work in this thesis gives rise to a number of new questions we would like the answers to. Some of these questions have already been posed as 'Open Questions' in previous chapters; we summarise below the content of these questions.

- Regarding the procedure outlined in Definition 4.5.1 for computing an Involutive Basis for a non-homogeneous basis by homogeneous methods, if the set G returned by the procedure is not autoreduced, under what circumstances does autoreducing G result in obtaining a set that is an Involutive Basis for the ideal generated by the input basis F?
- Apart from the empty, left and right divisions, are there any other global noncommutative involutive divisions of the following types:
  (a) strong and continuous;
  (b) weak, continuous and Gröbner?
- Are there any conclusive noncommutative involutive divisions that are also continuous and either strong or Gröbner?
- Is it possible to perform a noncommutative walk between two admissible and extendible monomial orderings that are not harmonious?

In addition to seeking answers to the above questions, there are a number of other directions we could take. One area to explore would be the development of the algorithms introduced in this thesis. For example, can the improvements made to the involutive algorithms in the commutative case, such as the *a priori* detection of prolongations that involutively reduce to zero (see [23]), be applied to the noncommutative case? Also, can we develop multiple-object versions of our algorithms, so that (for example) noncommutative Involutive Bases for path algebras can be computed?

Implementations of any new or improved algorithms would clearly build upon the code presented in Appendix B. We could also expand this code by implementing logged versions of our algorithms; implementing efficient methods for performing involutive reduction (as seen for example in Section 5.8.1); and implementing the algorithms from Chapter 6

for performing noncommutative walks. These improved algorithms and implementations could then be used (perhaps) to help judge the relative efficiency and complexity of the involutive methods versus the Gröbner methods.

Applications

As every noncommutative Involutive Basis is a noncommutative Gröbner Basis (at least for the involutive divisions defined in this thesis), applications for noncommutative Involutive Bases will mirror those for noncommutative Gröbner Bases. Some areas in which noncommutative Gröbner Bases have already been used include operator theory; systems engineering and linear control theory [32]. Other areas in noncommutative algebra which could also benefit from the theory introduced in this thesis include term rewriting; Petri nets; linear logic; quantum groups and coherence problems.

Further applications may come if we can extend our algorithms to the multiple-object case. It would be interesting (for example) to compare a multiple-object algorithm to a (standard) one-object algorithm in cases where an Involutive Basis for a multiple-object example can be computed using the one-object algorithm by adding some extra relations. This would tie in nicely with the existing comparison between the commutative and noncommutative versions of the Gröbner Basis algorithm, where it has been noticed that although commutative examples can be computed using the noncommutative algorithm, taking this route may in fact be less efficient than when using the commutative algorithm to do the same computation.

Appendix A

Proof of Propositions 5.5.31 and 5.5.32

A.1 Proposition 5.5.31

(Proposition 5.5.31) The two-sided left overlap division $\mathcal{W}$ is continuous.

Proof: Let w be an arbitrary fixed monomial; let U be any set of monomials; and consider any sequence $(u_1,\ u_2,\ \ldots,\ u_k)$ of monomials from U ($u_i \in U$ for all $1 \leqslant i \leqslant k$), each of which is a conventional divisor of w (so that $w = \ell_i u_i r_i$ for all $1 \leqslant i \leqslant k$, where the ℓ_i and the r_i are monomials). For all $1 \leqslant i < k$, suppose that the monomial u_{i+1} satisfies exactly one of the conditions (a) and (b) from Definition 5.4.2 (where multiplicative variables are taken with respect to $\mathcal{W}$ over the set U). To show that $\mathcal{W}$ is continuous, we must show that no two pairs (ℓ_i, r_i) and (ℓ_j, r_j) are the same, where $i \neq j$.

Assume to the contrary that there are at least two identical pairs in the sequence

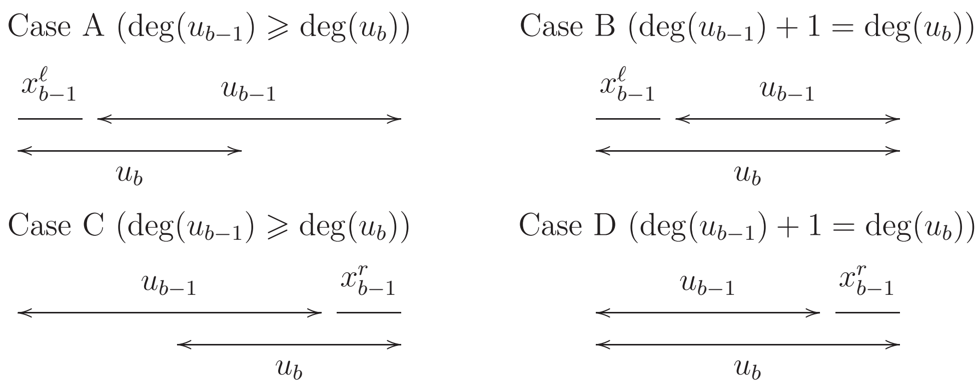
$$((\ell_1, r_1),\ (\ell_2, r_2),\ \ldots,\ (\ell_k, r_k)),$$

so that we can choose two separate pairs (ℓ_a, r_a) and (ℓ_b, r_b) from this sequence such that $(\ell_a, r_a) = (\ell_b, r_b)$ and all the pairs (ℓ_c, r_c) (for $a \leqslant c < b$) are different. We will now show that such a sequence $((\ell_a, r_a),\ \ldots,\ (\ell_b, r_b))$ cannot exist.

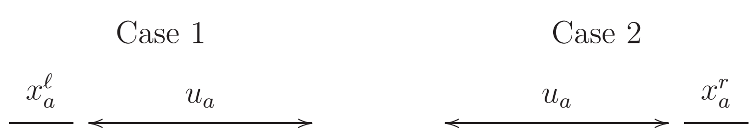
To begin with, notice that for each monomial u_{i+1} in the sequence $(u_1, \ldots, u_k)$ of mono-

mials ($1 \leqslant i < k$), if u_{i+1} involutively divides a left prolongation of the monomial u_i (so that $u_{i+1} \mid_{\mathcal{W}} (\text{Suffix}(\ell_i, 1))u_i$), then u_{i+1} must be a prefix of this prolongation; if u_{i+1} involutively divides a right prolongation of the monomial u_i (so that $u_{i+1} \mid_{\mathcal{W}} u_i(\text{Prefix}(r_i, 1))$), then u_{i+1} must be a suffix of this prolongation. This is because in all other cases, u_{i+1} is either equal to u_i, in which case u_{i+1} cannot involutively divide the (left or right) prolongation of u_i trivially; or u_{i+1} is a subword of u_i, in which case u_{i+1} cannot involutively divide the (left or right) prolongation of u_i by definition of $\mathcal{W}$.

Following on from the above, we can deduce that u_b is either a suffix or a prefix of a prolongation of u_{b-1}, leaving the following four cases, where $x^\ell_{b-1} = \text{Suffix}(\ell_{b-1}, 1)$ and $x^r_{b-1} = \text{Prefix}(r_{b-1}, 1)$.



These four cases can all originate from one of the following two cases (starting with a left prolongation or a right prolongation), where $x^\ell_a = \text{Suffix}(\ell_a, 1)$ and $x^r_a = \text{Prefix}(r_a, 1)$.

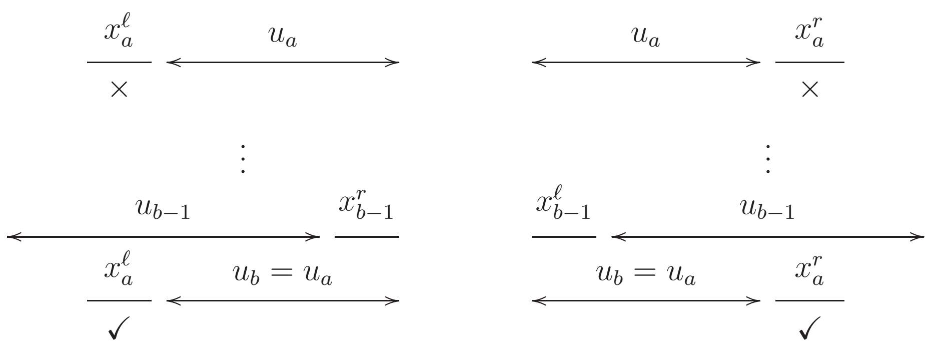


So there are eight cases to deal with in total, namely cases 1-A, 1-B, 1-C, 1-D, 2-A, 2-B, 2-C and 2-D.

We can immediately rule out cases 1-C and 2-A because we can show that a particular variable is both multiplicative and nonmultiplicative for monomial $u_a = u_b$ with respect to U, a contradiction. In case 1-C, the variable is x^ℓ_a: it has to be left nonmultiplicative to provide a left prolongation for u_a, and left multiplicative so that u_b is an involutive divisor of the right prolongation of u_{b-1}; in case 2-A, the variable is x^r_a: it has to be right nonmultiplicative to provide a right prolongation for u_a, and right multiplicative

so that u_b is an involutive divisor of the left prolongation of u_{b-1}. We illustrate this in the following diagrams by using a tick to denote a multiplicative variable and a cross to denote a nonmultiplicative variable.

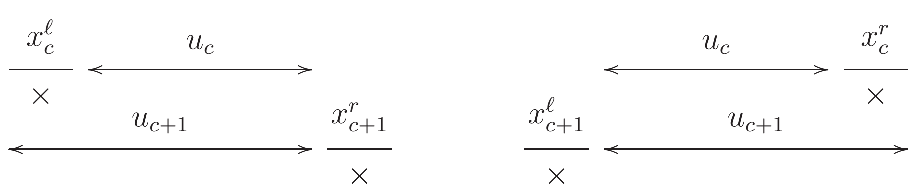
Case 1-C Case 2-A



For all the remaining cases, let us now consider how we may construct a sequence $((\ell_a, r_a), \ldots, (\ell_b, r_b) = (\ell_a, r_a))$. Because we know that each u_{c+1} is a prefix (or suffix) of a left (or right) prolongation of u_c (where $a \leqslant c < b$), it is clear that at some stage during the sequence, some u_{c+1} must be a proper suffix (or prefix) of a prolongation, or else the degrees of the monomials in the sequence $(u_a, \ldots)$ will strictly increase, meaning that we can never encounter the same (ℓ, r) pair twice. Further, the direction in which prolongations are taken must change some time during the sequence, or else the degrees of the monomials in one of the sequences $(\ell_a, \ldots)$ and $(r_a, \ldots)$ will strictly decrease, again meaning that we can never encounter the same (ℓ, r) pair twice.

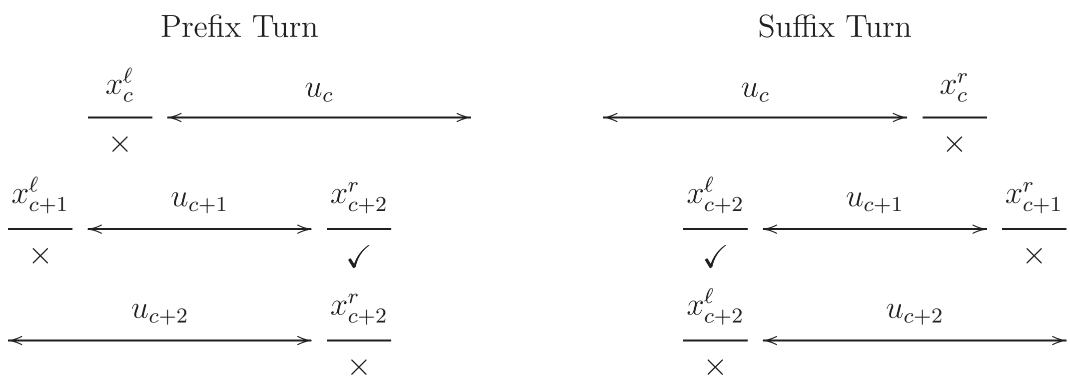
A change in direction can only occur if u_{c+1} is equal to a prolongation of u_c, as illustrated below.

Left Prolongation Turn Right Prolongation Turn



However, if no proper prefixes or suffixes are taken during the sequence, it is clear that making left or right prolongation turns will not affect the fact that the degrees of the monomials in the sequence $(u_a, \ldots)$ will strictly increase, once again meaning that we can never encounter the same (ℓ, r) pair twice. It follows that our only course of action is to

make a (left or right) prolongation turn after a proper prefix or a suffix of a prolongation has been taken. We shall call such prolongation turns *prefix* or *suffix turns*.



Claim: It is impossible to perform a prefix turn when $\mathcal{W}$ has been used to assign multiplicative variables.

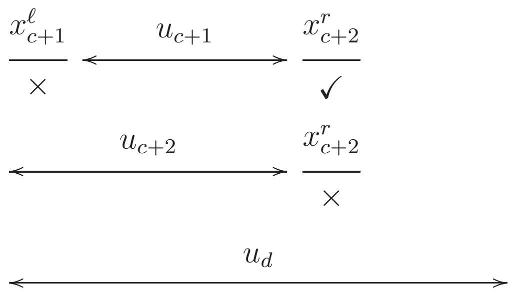
Proof of Claim: It is sufficient to show that $\mathcal{W}$ cannot assign multiplicative variables to U as follows:

$$x_c^\ell \notin \mathcal{M}^L_{\mathcal{W}}(u_c, U);\ x_{c+2}^r \in \mathcal{M}^R_{\mathcal{W}}(u_{c+1}, U);\ x_{c+2}^r \notin \mathcal{M}^R_{\mathcal{W}}(u_{c+2}, U). \tag{A.1}$$

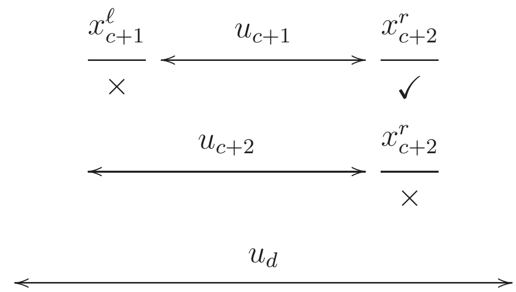
Consider how Algorithm 16 can assign the variable x_{c+2}^r to be right nonmultiplicative for monomial u_{c+2}. As things are set up in the digram for the prefix turn, the only possibility is that it is assigned due to the shown overlap between u_c and u_{c+2}. But this assumes that these two monomials actually overlap (which won't be the case if $\deg(u_{c+1}) = 1$); that u_c is greater than or equal to u_{c+2} with respect to the DegRevLex monomial ordering (so any overlap assigns a nonmultiplicative variable to u_{c+2}, not to u_c); and that, by the time we come to consider the prefix overlap between u_c and u_{c+2} in Algorithm 16, the variable x_c^ℓ must be left multiplicative for monomial u_c. But this final condition ensures that Algorithm 16 will terminate with x_c^ℓ being left multiplicative for u_c, contradicting Equation (A.1). We therefore conclude that the variable x_{c+2}^r must be assigned right nonmultiplicative for monomial u_{c+2} via some other overlap.

There are three possibilities for this overlap: (i) there exists a monomial $u_d \in U$ such that u_{c+2} is a prefix of u_d; (ii) there exists a monomial $u_d \in U$ such that u_{c+2} is a subword of u_d; and (iii) there exists a monomial $u_d \in U$ such that some prefix of u_d is equal to some suffix of u_{c+2}.

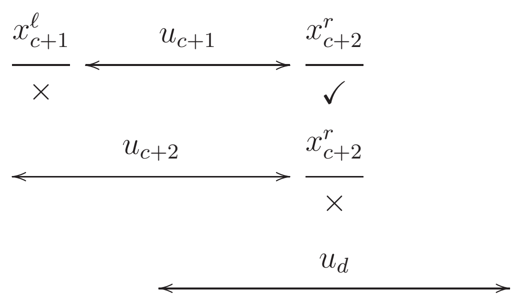
Overlap (i)



Overlap (ii)



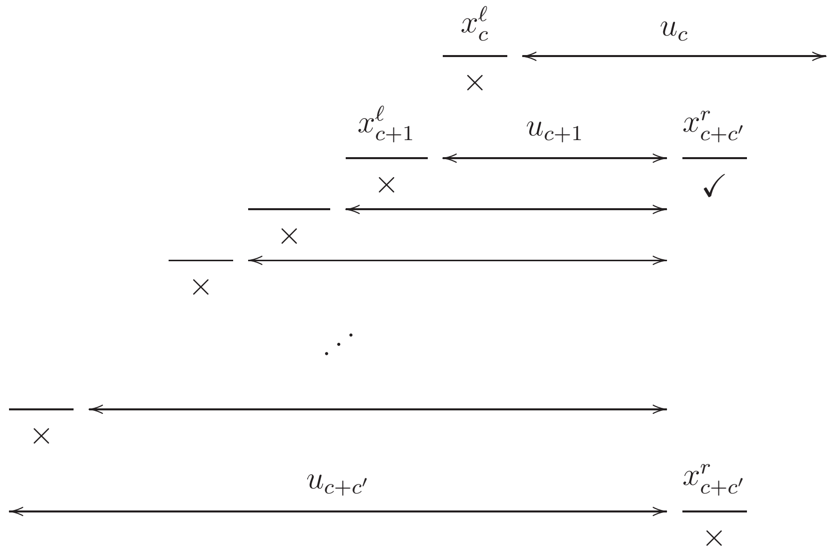
Overlap (iii)



In cases (i) and (ii), the overlap shown between u_{c+1} and u_d ensures that Algorithm 16 will always assign x^r_{c+2} to be right nonmultiplicative for monomial u_{c+1}, contradicting Equation (A.1). This leaves case (iii), which we break down into two further subcases, dependent upon whether u_{c+1} is a prefix of u_d or not. If u_{c+1} is a prefix of u_d, then Algorithm 16 will again assign x^r_{c+2} to be right nonmultiplicative for u_{c+1}, contradicting Equation (A.1). Otherwise, assuming that the shown overlap between u_{c+2} and u_d assigns x^r_{c+2} to be right nonmultiplicative for u_{c+2} (so that the variable immediately to the left of monomial u_d must be left multiplicative), we must again come to the conclusion that variable x^r_{c+2} is right nonmultiplicative for u_{c+1} (due to the overlap between u_{c+1} and u_d), once again contradicting Equation (A.1).

Technical Point: It is possible that several left prolongations may occur between the monomials u_{c+1} and u_{c+2} shown in the diagram for the prefix turn, but, as long as no proper prefixes are taken during this sequence (in which case we potentially start another prefix turn), we can apply the same proof as above (replacing $c+2$ by $c+c'$) to show that we cannot perform an *extended* prefix turn (as shown below) with respect to $\mathcal{W}$.

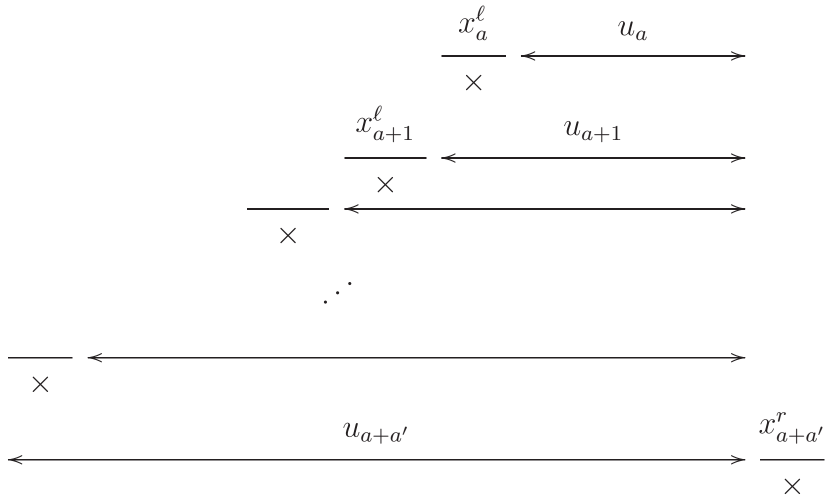
Extended Prefix Turn



$\square$

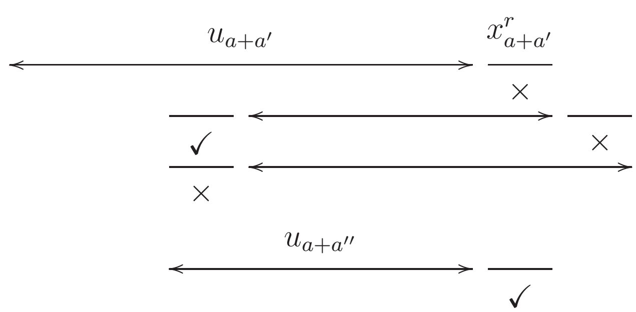
Having ruled out prefix turns, we can now eliminate cases 1-D, 2-C and 2-D because they require (i) a proper prefix to be taken during the sequence (allowing $\deg(r_{b-1}) = \deg(r_b) + 1$); and (ii) the final prolongation to be a right prolongation, ensuring that a turn has to follow the proper prefix, and so an (extended) prefix turn is required.

For Cases 1-A and 1-B, we start by taking a left prolongation, which means that somewhere during the sequence a proper suffix must be taken. To do this, it follows that we must change the direction that prolongations are taken. Knowing that prefix turns are ruled out, we must therefore turn by using a left prolongation turn, which will happen after a finite number $a' \geqslant 1$ of left prolongations.



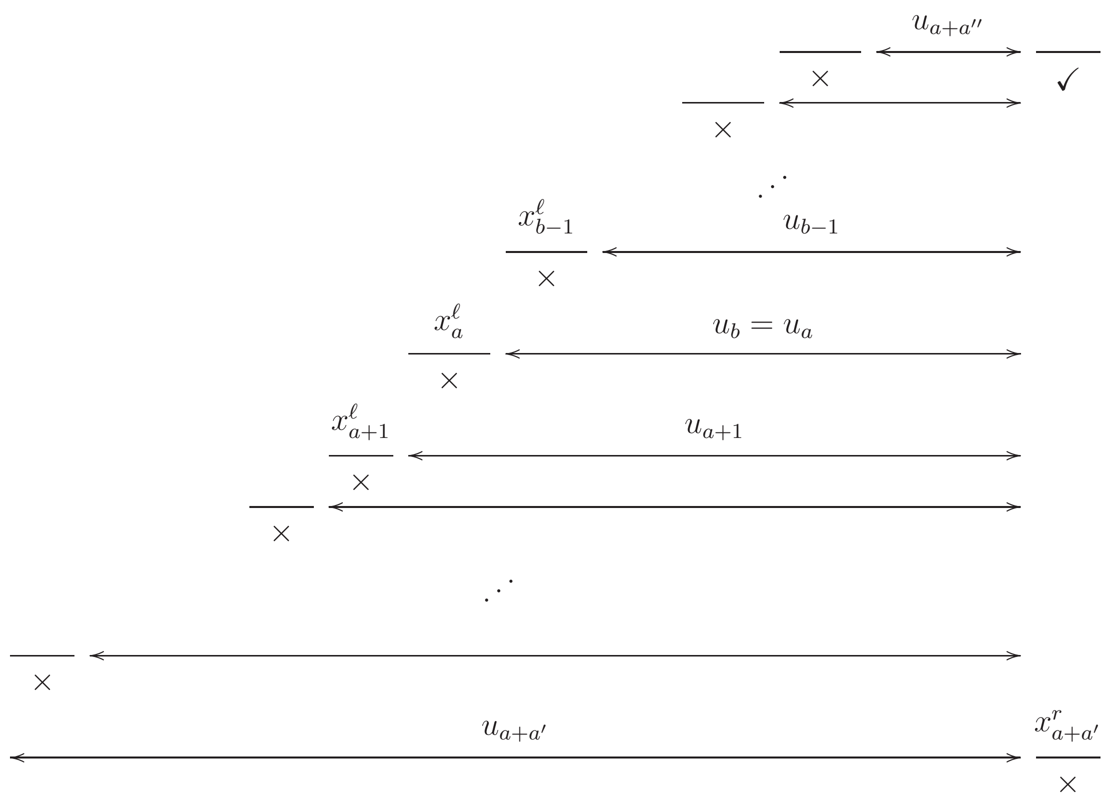
Considering how Algorithm 16 assigns the variable $x_{a+a'}^r$ to be right nonmultiplicative for monomial $u_{a+a'}$, there are three possibilities: (i) there exists a monomial $u_d \in U$ such that $u_{a+a'}$ is a prefix of u_d; (ii) there exists a monomial $u_d \in U$ such that $u_{a+a'}$ is a subword of u_d; and (iii) there exists a monomial $u_d \in U$ such that some prefix of u_d is equal to some suffix of $u_{a+a'}$. In each of these cases, there will be an overlap between u_a and u_d that will ensure that Algorithm 16 also assigns the variable $x_{a+a'}^r$ to be right nonmultiplicative for monomial u_a. This rules out Case 1-A, as variable $x_{a+a'}^r$ must be right multiplicative for monomial $u_b = u_a$ in order to perform the final step of Case 1-A.

For Case 1-B, we must now make an (extended) suffix turn as we need to finish the sequence prolongating to the left. But, once we have done this, we must subsequently take a proper prefix in order to ensure that u_{b-1} is a suffix of $u_a = u_b$. Pictorially, here is one way of accomplishing this, where we note that any number of prolongations may occur between any of the shown steps.



Once we have reached the stage where we are working with a suffix of u_a, we may continue prolongating to the left until we form the monomial $u_b = u_a$, seemingly providing a

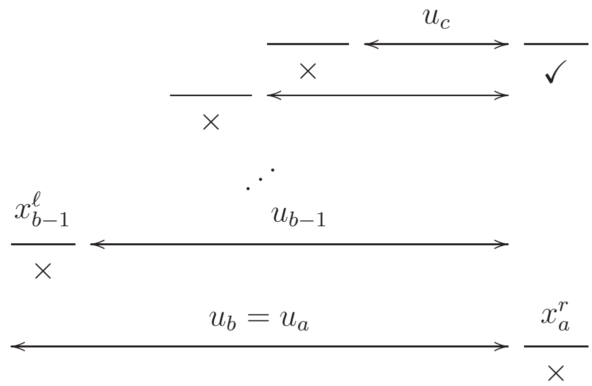
counterexample to the proposition (we have managed to construct the same (ℓ, r) pair twice). However, starting with the monomial labelled $u_{a+a''}$ in the above diagram, if we follow the sequence from $u_{a+a''}$ via left prolongations to $u_b = u_a$, and then continue with the same sequence as we started off with, we notice that by the time we encounter the monomial $u_{a+a'}$ again, an extended prefix turn has been made, in effect meaning that the first prolongation of u_a we took right at the start of the sequence was invalid.



This leaves Case 2-B. Here we start by taking a right prolongation, meaning that somewhere during the sequence a proper prefix must be taken. To do this, it follows that we must change the direction that prolongations are taken. There are two ways of doing this: (i) by using an (extended) suffix turn; (ii) by using a right prolongation turn.

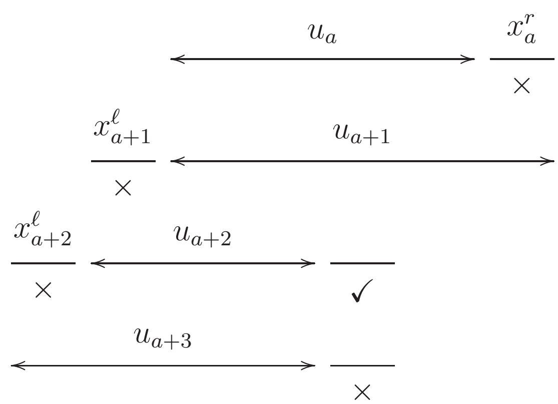
In case (i), after performing the (extended) suffix turn, we need to take a proper prefix so that the next monomial (say u_c) in the sequence is a suffix of u_a; we then continue by taking left prolongations until we form the monomial $u_b = u_a$. This provides an apparent counterexample to the proposition, but as for Case 1-B above, by taking the right prolongation of u_a the second time around, we perform an extended prefix turn, rendering the *first* right prolongation of u_a invalid.

Case (i)



In case (ii), after we make a right prolongation turn (which may itself occur after a finite number of right prolongations), we may now take the required proper prefix. But as we are then required to take a proper suffix (in order to ensure that we finish the sequence taking a left prolongation), we need to make a turn. But as this would entail making an (extended) prefix turn, we conclude that case (ii) is also invalid.

An Example of Case (ii)



As we have now accounted for all eight possible sequences, we can conclude that $\mathcal{W}$ is continuous. $\square$

A.2 Proposition 5.5.32

(Proposition 5.5.32) The two-sided left overlap division $\mathcal{W}$ is a Gröbner involutive division.

Proof: We are required to show that if Algorithm 12 terminates with $\mathcal{W}$ and some arbitrary admissible monomial ordering O as input, then the Locally Involutive Basis G it returns is a noncommutative Gröbner Basis. By Definition 3.1.8, we can do this by showing that all S-polynomials involving elements of G conventionally reduce to zero using G.

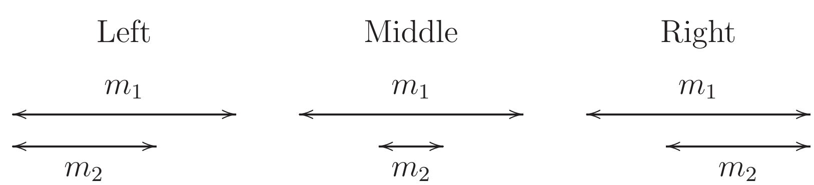
Assume that $G = \{g_1, \dots, g_p\}$ is sorted (by lead monomial) with respect to the DegRevLex monomial ordering (greatest first), and let $U = \{u_1, \dots, u_p\} := \{\mathrm{LM}(g_1), \dots, \mathrm{LM}(g_p)\}$ be the set of leading monomials. Let T be the table obtained by applying Algorithm 16 to U. Because G is a Locally Involutive Basis, every zero entry $T(u_i, x_j^\Gamma)$ ($\Gamma \in \{L, R\}$) in the table corresponds to a prolongation $g_i x_j$ or $x_j g_i$ that involutively reduces to zero.

Let S be the set of S-polynomials involving elements of G, where the t-th entry of S ($1 \leqslant t \leqslant |S|$) is the S-polynomial

$$s_t = c_t \ell_t g_i r_t - c'_t \ell'_t g_j r'_t,$$

with $\ell_t u_i r_t = \ell'_t u_j r'_t$ being the overlap word of the S-polynomial. We will prove that every S-polynomial in S conventionally reduces to zero using G.

Recall (from Definition 3.1.2) that each S-polynomial in S corresponds to a particular type of overlap — 'prefix', 'subword' or 'suffix'. For the purposes of this proof, let us now split the subword overlaps into three further types — 'left', 'middle' and 'right', corresponding to the cases where a monomial m_2 is a prefix, proper subword and suffix of a monomial m_1.



This classification provides us with five cases to deal with in total, which we shall process in the following order: right, middle, left, prefix, suffix.

(1) Consider an arbitrary entry $s_t \in S$ $(1 \leqslant t \leqslant |S|)$ corresponding to a right overlap where the monomial u_j is a suffix of the monomial u_i. This means that $s_t = c_t g_i - c'_t \ell'_t g_j$ for some $g_i, g_j \in G$, with overlap word $u_i = \ell'_t u_j$. Let $u_i = x_{i_1} \ldots x_{i_\alpha}$; let $u_j = x_{j_1} \ldots x_{j_\beta}$; and let $D = \alpha - \beta$.

$$
\begin{array}{lcccccccccc}
u_i = & x_{i_1} & x_{i_2} & --- & x_{i_D} & x_{i_{D+1}} & x_{i_{D+2}} & --- & x_{i_{\alpha-1}} & x_{i_\alpha} \\
u_j = & & & & & x_{j_1} & x_{j_2} & --- & x_{j_{\beta-1}} & x_{j_\beta}
\end{array}
$$

Because u_j is a suffix of u_i, it follows that $T(u_j, x^L_{i_D}) = 0$. This gives rise to the prolongation $x_{i_D} g_j$ of g_j. But we know that all prolongations involutively reduce to zero (G is a Locally Involutive Basis), so Algorithm 10 must find a monomial $u_k = x_{k_1} \ldots x_{k_\gamma} \in U$ such that u_k involutively divides $x_{i_D} u_j$. Assuming that $x_{k_\gamma} = x_{i_\kappa}$, we can deduce that any candidate for u_k must be a suffix of $x_{i_D} u_j$ (otherwise $T(u_k, x^R_{i_{\kappa+1}}) = 0$ because of the overlap between u_i and u_k). But if u_k is a suffix of $x_{i_D} u_j$, then we must have $u_k = x_{i_D} u_j$ (otherwise $T(u_k, x^L_{i_{\alpha-\gamma}}) = 0$ again because of the overlap between u_i and u_k). We have therefore shown that there exists a monomial $u_k = x_{k_1} \ldots x_{k_\gamma} \in U$ such that u_k is a suffix of u_i and $\gamma = \beta + 1$.

$$
\begin{array}{lcccccccccc}
u_i = & x_{i_1} & x_{i_2} & --- & x_{i_D} & x_{i_{D+1}} & x_{i_{D+2}} & --- & x_{i_{\alpha-1}} & x_{i_\alpha} \\
u_j = & & & & & x_{j_1} & x_{j_2} & --- & x_{j_{\beta-1}} & x_{j_\beta} \\
u_k = & & & & x_{k_1} & x_{k_2} & x_{k_3} & --- & x_{k_{\gamma-1}} & x_{k_\gamma}
\end{array}
$$

In the case $D = 1$, it is clear that $u_k = u_i$, and so the first step in the involutive reduction of the prolongation $x_{i_1} g_j$ of g_j is to take away the multiple $(\frac{c_t}{c'_t}) g_i$ of g_i from $x_{i_1} g_j$ to leave the polynomial $x_{i_1} g_j - (\frac{c_t}{c'_t}) g_i = -(\frac{1}{c'_t}) s_t$. But as we know that all prolongations involutively reduce to zero, we can conclude that the S-polynomial s_t conventionally reduces to zero.

For the case $D > 1$, we can use the monomial u_k together with Buchberger's Second Criterion to simplify our goal of showing that the S-polynomial s_t reduces to zero. Notice that the monomial u_k is a subword of the overlap word u_i associated to s_t, and so in order to show that s_t reduces to zero, all we have to do is to show that the two S-polynomials

$$s_u = c_u g_i - c'_u (x_{i_1} x_{i_2} \ldots x_{i_{D-1}}) g_k$$

and

$$s_v = c_v g_k - c'_v x_{i_D} g_j$$

reduce to zero ($1 \leqslant u, v \leqslant |S|$). But s_v is an S-polynomial corresponding to a right overlap of type $D = 1$ (because $\gamma - \beta = 1$), and so s_v reduces to zero. It remains to show that the S-polynomial s_u reduces to zero. But we can do this by using exactly the same argument as above — we can show that there exists a monomial $u_\pi = x_{\pi_1} \dots x_{\pi_\delta} \in U$ such that u_π is a suffix of u_i and $\delta = \gamma + 1$, and we can deduce that the S-polynomial s_u reduces to zero (and hence s_t reduces to 0) if the S-polynomial

$$s_w = c_w g_i - c'_w (x_{i_1} x_{i_2} \dots x_{i_{D-2}}) g_\pi$$

reduces to zero ($1 \leqslant w \leqslant |S|$). By induction, there is a sequence $\{u_{q_D}, u_{q_{D-1}}, \dots, u_{q_2}\}$ of monomials increasing uniformly in degree, so that s_t reduces to zero if the S-polynomial

$$s_\eta = c_\eta g_i - c'_\eta x_{i_1} g_{q_2}$$

reduces to zero ($1 \leqslant \eta \leqslant |S|$).

$$\begin{array}{rcccccccccc}
u_i = & \underline{\quad x_{i_1} \quad} & \underline{\quad x_{i_2} \quad} & - - - & \underline{\quad x_{i_{D-1}} \quad} & \underline{\quad x_{i_D} \quad} & \underline{\quad x_{i_{D+1}} \quad} & \underline{\quad x_{i_{D+2}} \quad} & - - - & \underline{\quad x_{i_{\alpha-1}} \quad} & \underline{\quad x_{i_\alpha} \quad} \\
u_j = & & & & & & \underline{\quad x_{j_1} \quad} & \underline{\quad x_{j_2} \quad} & - - - & \underline{\quad x_{j_{\beta-1}} \quad} & \underline{\quad x_{j_\beta} \quad} \\
u_{q_D} = u_k = & & & & & \underline{\qquad} & \underline{\qquad} & \underline{\qquad} & - - - & \underline{\qquad} & \underline{\qquad} \\
u_{q_{D-1}} = u_\pi = & & & & \underline{\qquad} & \underline{\qquad} & \underline{\qquad} & \underline{\qquad} & - - - & \underline{\qquad} & \underline{\qquad} \\
\vdots & & & & & & & \iddots & & & \\
u_{q_2} = & & \underline{\qquad} & - - - & \underline{\qquad} & \underline{\qquad} & \underline{\qquad} & \underline{\qquad} & - - - & \underline{\qquad} & \underline{\qquad}
\end{array}$$

But s_η is always an S-polynomial corresponding to a right overlap of type $D = 1$, and so s_η reduces to zero — meaning we can conclude that s_t reduces to zero as well.

(2) Consider an arbitrary entry $s_t \in S$ ($1 \leqslant t \leqslant |S|$) corresponding to a middle overlap where the monomial u_j is a proper subword of the monomial u_i. This means that $s_t = c_t g_i - c'_t \ell'_t g_j r'_t$ for some $g_i, g_j \in G$, with overlap word $u_i = \ell'_t u_j r'_t$. Let $u_i = x_{i_1} \dots x_{i_\alpha}$; let $u_j = x_{j_1} \dots x_{j_\beta}$; and choose D such that $x_{i_D} = x_{j_\beta}$.

$$\begin{array}{rccccccccccc}
u_i = & \underline{\quad x_{i_1} \quad} & - - - & \underline{\quad x_{i_{D-\beta}} \quad} & \underline{\quad x_{i_{D-\beta+1}} \quad} & \underline{\quad x_{i_{D-\beta+2}} \quad} & - - - & \underline{\quad x_{i_{D-1}} \quad} & \underline{\quad x_{i_D} \quad} & \underline{\quad x_{i_{D+1}} \quad} & - - - & \underline{\quad x_{i_\alpha} \quad} \\
u_j = & & & & \underline{\quad x_{j_1} \quad} & \underline{\quad x_{j_2} \quad} & - - - & \underline{\quad x_{j_{\beta-1}} \quad} & \underline{\quad x_{j_\beta} \quad} & & &
\end{array}$$

Because u_j is a proper subword of u_i, it follows that $T(u_j, x^R_{i_{D+1}}) = 0$. This gives rise to

the prolongation $g_j x_{i_{D+1}}$ of g_j. But we know that all prolongations involutively reduce to zero, so there must exist a monomial $u_k = x_{k_1} \dots x_{k_\gamma} \in U$ such that u_k involutively divides $u_j x_{i_{D+1}}$. Assuming that $x_{k_\gamma} = x_{i_\kappa}$, any candidate for u_k must be a suffix of $u_j x_{i_{D+1}}$ (otherwise $T(u_k, x^R_{i_{\kappa+1}}) = 0$ because of the overlap between u_i and u_k). Unlike part (1) however, we cannot determine the degree of u_k (so that $1 \leqslant \gamma \leqslant \beta+1$); we shall illustrate this in the following diagram by using a squiggly line to indicate that the monomial u_k can begin anywhere (or nowhere if $u_k = x_{i_{D+1}}$) on the squiggly line.

$$
\begin{array}{lcccccccccccc}
u_i = & \overline{x_{i_1}} & --- & \overline{x_{i_{D-\beta}}} & \overline{x_{i_{D-\beta+1}}} & \overline{x_{i_{D-\beta+2}}} & --- & \overline{x_{i_{D-1}}} & \overline{x_{i_D}} & \overline{x_{i_{D+1}}} & --- & \overline{x_{i_\alpha}} \\
u_j = & & & & \overline{x_{j_1}} & \overline{x_{j_2}} & --- & \overline{x_{j_{\beta-1}}} & \overline{x_{j_\beta}} & & & \\
u_k = & & & & & & & & & \overline{x_{k_\gamma}} & &
\end{array}
$$

We can now use the monomial u_k together with Buchberger's Second Criterion to simplify our goal of showing that the S-polynomial s_t reduces to zero. Notice that the monomial u_k is a subword of the overlap word u_i associated to s_t, and so in order to show that s_t reduces to zero, all we have to do is to show that the two S-polynomials

$$s_u = c_u g_i - c'_u (x_{i_1} x_{i_2} \dots x_{i_{D+1-\gamma}}) g_k (x_{i_{D+2}} \dots x_{i_\alpha})$$

and[1]

$$s_v = c_v (x_{j_1} \dots x_{i_{D+1-\gamma}}) g_k - c'_v g_j x_{i_{D+1}}$$

reduce to zero ($1 \leqslant u, v \leqslant |S|$).

For the S-polynomial s_v, there are two cases to consider: $\gamma = 1$, and $\gamma > 1$. In the former case, because (as placed in u_i) the monomials u_j and u_k do not overlap, we can use Buchberger's First Criterion to say that the 'S-polynomial' s_v reduces to zero (for further explanation, see the paragraph at the beginning of Section 3.4.1). In the latter case, note that u_k is the only involutive divisor of the prolongation $u_j x_{i_{D+1}}$, as the existence of any suffix of $u_j x_{i_{D+1}}$ of higher degree than u_k in U will contradict the fact that u_k is an involutive divisor of $u_j x_{i_{D+1}}$; and the existence of u_k in U ensures that any suffix of $u_j x_{i_{D+1}}$ that exists in U with a lower degree than u_k will not be an involutive divisor of $u_j x_{i_{D+1}}$. This means that the first step of the involutive reduction of $g_j x_{i_{D+1}}$ is to take away the multiple $(\frac{c_v}{c'_v})(x_{j_1} \dots x_{i_{D+1-\gamma}}) g_k$ of g_k from $g_j x_{i_{D+1}}$ to leave the polynomial $g_j x_{i_{D+1}} -$

[1]Technical point: if $\gamma \neq \beta + 1$, the S-polynomial s_v could in fact appear as $s_v = c_v g_j x_{i_{D+1}} - c'_v (x_{j_1} \dots x_{i_{D+1-\gamma}}) g_k$ and not as $s_v = c_v (x_{j_1} \dots x_{i_{D+1-\gamma}}) g_k - c'_v g_j x_{i_{D+1}}$; for simplicity we will treat both cases the same in the proof as all that changes is the notation and the signs.

$(\frac{c_v}{c'_v})(x_{j_1}\dots x_{i_{D+1-\gamma}})g_k = -(\frac{1}{c'_v})s_v$. But as we know that all prolongations involutively reduce to zero, we can conclude that the S-polynomial s_v conventionally reduces to zero.

For the S-polynomial s_u, we note that if $D = \alpha - 1$, then s_u corresponds to a right overlap, and so we know from part (1) that s_u conventionally reduces to zero. Otherwise, we proceed by induction on the S-polynomial s_u to produce a sequence $\{u_{q_{D+1}}, u_{q_{D+2}}, \dots, u_{q_\alpha}\}$ of monomials, so that s_u (and hence s_t) reduces to zero if the S-polynomial

$$s_\eta = c_\eta g_i - c'_\eta(x_{i_1}\dots x_{i_{\alpha-\mu}})g_{q_\alpha}$$

reduces to zero ($1 \leqslant \eta \leqslant |S|$), where $\mu = \deg(u_{q_\alpha})$.

$$\begin{array}{rl}
u_i = & x_{i_1} - - - \; x_{i_{D-\beta}} \; x_{i_{D-\beta+1}} - - - \; x_{i_D} \; x_{i_{D+1}} \; x_{i_{D+2}} - - - \; x_{i_{\alpha-1}} \; x_{i_\alpha} \\
u_j = & x_{j_1} - - - \; x_{j_\beta} \\
u_{q_{D+1}} = u_k = & \sim\sim\sim\sim\sim\sim \; x_{k_\gamma} \\
u_{q_{D+2}} = & \sim\sim\sim\sim\sim\sim\sim\sim \; \rule{1em}{0.4pt} \\
 & \ddots \\
u_{q_\alpha} = & \sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim \; \rule{1em}{0.4pt}
\end{array}$$

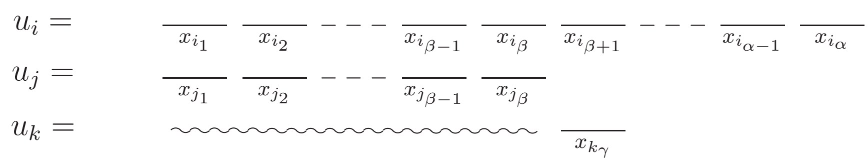
But s_η always corresponds to a right overlap, and so s_η reduces to zero — meaning we can conclude that s_t reduces to zero as well.

(3) Consider an arbitrary entry $s_t \in S$ ($1 \leqslant t \leqslant |S|$) corresponding to a left overlap where the monomial u_j is a prefix of the monomial u_i. This means that $s_t = c_t g_i - c'_t g_j r'_t$ for some $g_i, g_j \in G$, with overlap word $u_i = u_j r'_t$. Let $u_i = x_{i_1}\dots x_{i_\alpha}$ and let $u_j = x_{j_1}\dots x_{j_\beta}$.

$$\begin{array}{rl}
u_i = & x_{i_1} \; x_{i_2} - - - \; x_{i_{\beta-1}} \; x_{i_\beta} \; x_{i_{\beta+1}} - - - \; x_{i_{\alpha-1}} \; x_{i_\alpha} \\
u_j = & x_{j_1} \; x_{j_2} - - - \; x_{j_{\beta-1}} \; x_{j_\beta}
\end{array}$$

Because u_j is a prefix of u_i, it follows that $T(u_j, x^R_{i_{\beta+1}}) = 0$. This gives rise to the prolongation $g_j x_{i_{\beta+1}}$ of g_j. But we know that all prolongations involutively reduce to zero, so there must exist a monomial $u_k = x_{k_1}\dots x_{k_\gamma} \in U$ such that u_k involutively divides $u_j x_{i_{\beta+1}}$. Assuming that $x_{k_\gamma} = x_{i_\kappa}$, any candidate for u_k must be a suffix of $u_j x_{i_{\beta+1}}$

(otherwise $T(u_k, x^R_{i_{\kappa+1}}) = 0$ because of the overlap between u_i and u_k).



If $\alpha = \gamma$, then it is clear that $u_k = u_i$, and so the first step in the involutive reduction of the prolongation $g_j x_{i_\alpha}$ is to take away the multiple $(\frac{c_t}{c'_t})g_i$ of g_i from $g_j x_{i_\alpha}$ to leave the polynomial $g_j x_{i_\alpha} - (\frac{c_t}{c'_t})g_i = -(\frac{1}{c'_t})s_t$. But as we know that all prolongations involutively reduce to zero, we can conclude that the S-polynomial s_t conventionally reduces to zero.

Otherwise, if $\alpha > \gamma$, we can now use the monomial u_k together with Buchberger's Second Criterion to simplify our goal of showing that the S-polynomial s_t reduces to zero. Notice that the monomial u_k is a subword of the overlap word u_i associated to s_t, and so in order to show that s_t reduces to zero, all we have to do is to show that the two S-polynomials

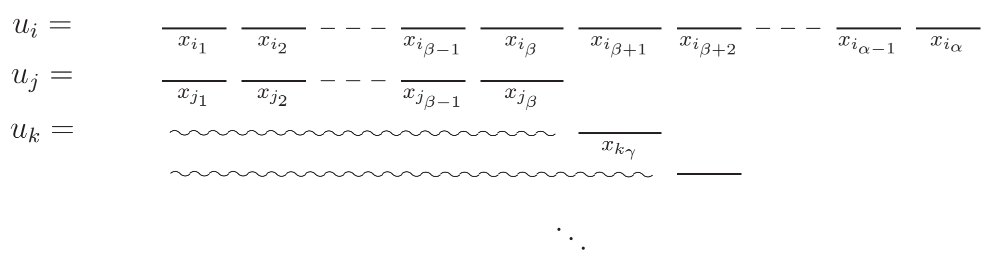
$$s_u = c_u g_i - c'_u(x_{i_1} \dots x_{i_{\beta+1-\gamma}})g_k(x_{i_{\beta+2}} \dots x_{i_\alpha})$$

and

$$s_v = c_v(x_{i_1} \dots x_{i_{\beta+1-\gamma}})g_k - c'_v g_j x_{i_{\beta+1}}$$

reduce to zero ($1 \leqslant u, v \leqslant |S|$).

The S-polynomial s_v reduces to zero by comparison with part (2). For the S-polynomial s_u, first note that if $\alpha = \beta + 1$, then s_u corresponds to a right overlap, and so we know from part (1) that s_u conventionally reduces to zero. Otherwise, if $\gamma \neq \beta + 1$, then s_u corresponds to a middle overlap, and so we know from part (2) that s_u conventionally reduces to zero. This leaves the case where s_u corresponds to another left overlap, in which case we proceed by induction on s_u, eventually coming across either a middle overlap or a right overlap because we move one letter at a time to the right after each inductive step.



(4 and 5) In Definition 3.1.2, we defined a prefix overlap to be an overlap where, given two monomials m_1 and m_2 such that $\deg(m_1) \geqslant \deg(m_2)$, a prefix of m_1 is equal to a suffix of m_2; suffix overlaps were defined similarly. If we drop the condition on the degrees of the monomials, it is clear that every suffix overlap can be treated as a prefix overlap (by swapping the roles of m_1 and m_2); this allows us to deal with the case of a prefix overlap only.

Consider an arbitrary entry $s_t \in S$ $(1 \leqslant t \leqslant |S|)$ corresponding to a prefix overlap where a prefix of the monomial u_i is equal to a suffix of the monomial u_j. This means that $s_t = c_t \ell_t g_i - c'_t g_j r'_t$ for some $g_i, g_j \in G$, with overlap word $\ell_t u_i = u_j r'_t$. Let $u_i = x_{i_1} \dots x_{i_\alpha}$; let $u_j = x_{j_1} \dots x_{j_\beta}$; and choose D such that $x_{i_D} = x_{j_\beta}$.

$$
\begin{array}{lcccccccccc}
u_i = & & & & \underline{}\,x_{i_1} & - - - & \underline{}\,x_{i_D} & \underline{}\,x_{i_{D+1}} & - - - & \underline{}\,x_{i_{\alpha-1}} & \underline{}\,x_{i_\alpha} \\
u_j = & \underline{}\,x_{j_1} & - - - & \underline{}\,x_{j_{\beta-D}} & \underline{}\,x_{j_{\beta-D+1}} & - - - & \underline{}\,x_{j_\beta} & & & &
\end{array}
$$

By definition of $\mathcal{W}$, at least one of $T(u_i, x^L_{j_{\beta-D}})$ and $T(u_j, x^R_{i_{D+1}})$ is equal to zero.

- Case $T(u_j, x^R_{i_{D+1}}) = 0$.

  Because we know that the prolongation $g_j x_{i_{D+1}}$ involutively reduces to zero, there must exist a monomial $u_k = x_{k_1} \dots x_{k_\gamma} \in U$ such that u_k involutively divides $u_j x_{i_{D+1}}$. This u_k must be a suffix of $u_j x_{i_{D+1}}$ (otherwise, assuming that $x_{k_\gamma} = x_{j_\kappa}$, we have $T(u_k, x^R_{i_{D+1}}) = 0$ if $\gamma = \beta$ (because of the overlap between u_i and u_k); $T(u_k, x^L_{j_{\beta-\gamma}}) = 0$ if $\gamma < \beta$ and $\kappa = \beta$ (because of the overlap between u_j and u_k); and $T(u_k, x^R_{j_{\kappa+1}}) = 0$ if $\gamma < \beta$ and $\kappa < \beta$ (again because of the overlap between u_j and u_k)).

$$
\begin{array}{lcccccccccc}
u_i = & & & & \underline{}\,x_{i_1} & - - - & \underline{}\,x_{i_D} & \underline{}\,x_{i_{D+1}} & - - - & \underline{}\,x_{i_{\alpha-1}} & \underline{}\,x_{i_\alpha} \\
u_j = & \underline{}\,x_{j_1} & - - - & \underline{}\,x_{j_{\beta-D}} & \underline{}\,x_{j_{\beta-D+1}} & - - - & \underline{}\,x_{j_\beta} & & & & \\
u_k = & \multicolumn{6}{c}{\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim} & \underline{}\,x_{k_\gamma} & & &
\end{array}
$$

  Let us now use the monomial u_k together with Buchberger's Second Criterion to simplify our goal of showing that the S-polynomial s_t reduces to zero. Because u_k is a subword of the overlap word $\ell_t u_i$ associated to s_t, in order to show that s_t reduces to zero, all we have to do is to show that the two S-polynomials

$$
s_u = \begin{cases} c_u(x_{k_1} \dots x_{j_{\beta-D}}) g_i - c'_u g_k (x_{i_{D+2}} \dots x_{i_\alpha}) & \text{if } \gamma > D+1 \\ c_u g_i - c'_u \ell'_u g_k (x_{i_{D+2}} \dots x_{i_\alpha}) & \text{if } \gamma \leqslant D+1 \end{cases}
$$

and

$$s_v = c_v g_j x_{i_{D+1}} - c'_v(x_{j_1} \dots x_{j_{\beta+1-\gamma}})g_k$$

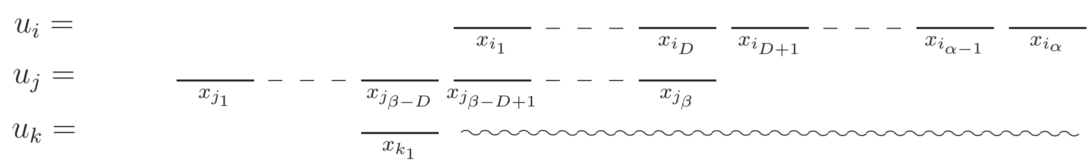
reduce to zero ($1 \leqslant u, v \leqslant |S|$).

The S-polynomial s_v reduces to zero by comparison with part (2). For the S-polynomial s_u, first note that if $\alpha = D+1$, then either u_k is a suffix of u_i, u_i is a suffix of u_k, or $u_k = u_i$; it follows that s_u reduces to zero trivially if $u_k = u_i$, and s_u reduces to zero by part (1) in the other two cases.

If however $\alpha \neq D+1$, then either s_u is a middle overlap (if $\gamma < D+1$), a left overlap (if $\gamma = D+1$), or another prefix overlap. The first two cases can be handled by parts (2) and (3) respectively; the final case is handled by induction, where we note that after each step of the induction, the value $\alpha + \beta - 2D$ strictly decreases (regardless of which case $T(u_j, x^R_{i_{D+1}}) = 0$ or $T(u_i, x^L_{j_{\beta-D}}) = 0$ applies), so we are guaranteed at some stage to find an overlap that is not a prefix overlap, enabling us to verify that the S-polynomial s_t conventionally reduces to zero.

- Case $T(u_i, x^L_{j_{\beta-D}}) = 0$.

Because we know that the prolongation $x_{j_{\beta-D}} g_i$ involutively reduces to zero, there must exist a monomial $u_k = x_{k_1} \dots x_{k_\gamma} \in U$ such that u_k involutively divides $x_{j_{\beta-D}} u_i$. This u_k must be a prefix of $x_{j_{\beta-D}} u_i$ (otherwise, assuming that $x_{k_\gamma} = x_{i_\kappa}$, we have $T(u_k, x^L_{j_{\beta-D}}) = 0$ if $\gamma = \alpha$ (because of the overlap between u_j and u_k); $T(u_k, x^L_{i_{\kappa-\gamma}}) = 0$ if $\gamma < \alpha$ and $\kappa = \alpha$ (because of the overlap between u_i and u_k); and $T(u_k, x^R_{i_{\kappa+1}}) = 0$ if $\gamma < \alpha$ and $\kappa < \alpha$ (again because of the overlap between u_i and u_k)).



Let us now use the monomial u_k together with Buchberger's Second Criterion to simplify our goal of showing that the S-polynomial s_t reduces to zero. Because u_k is a subword of the overlap word $\ell_t u_i$ associated to s_t, in order to show that s_t reduces to zero, all we have to do is to show that the two S-polynomials

$$s_u = c_u x_{k_1} g_i - c'_u g_k(x_{i_\gamma} \dots x_{i_\alpha})$$

and

$$s_v = \begin{cases} c_v g_j(x_{i_{D+1}} \dots x_{k_\gamma}) - c'_v(x_{j_1} \dots x_{j_{\beta-D-1}})g_k & \text{if } \gamma > D+1 \\ c_v g_j - c'_v(x_{j_1} \dots x_{j_{\beta-D-1}})g_k r'_v & \text{if } \gamma \leqslant D+1 \end{cases}$$

reduce to zero ($1 \leqslant u, v \leqslant |S|$).

The S-polynomial s_u reduces to zero by comparison with part (2). For the S-polynomial s_v, first note that if $\beta - D = 1$, then either u_k is a prefix of u_j, u_j is a prefix of u_k, or $u_k = u_j$; it follows that s_v reduces to zero trivially if $u_k = u_j$, and s_v reduces to zero by part (3) in the other two cases.

If however $\beta - D \neq 1$, then either s_v is a middle overlap (if $\gamma < D+1$), a right overlap (if $\gamma = D+1$), or another prefix overlap. The first two cases can be handled by parts (2) and (1) respectively; the final case is handled by induction, where we note that after each step of the induction, the value $\alpha + \beta - 2D$ strictly decreases (regardless of which case $T(u_j, x^R_{i_{D+1}}) = 0$ or $T(u_i, x^L_{j_{\beta-D}}) = 0$ applies), so we are guaranteed at some stage to find an overlap that is not a prefix overlap, enabling us to verify that the S-polynomial s_t conventionally reduces to zero.

$\square$

Appendix B

Source Code

In this Appendix, we will present ANSI C source code for an initial implementation of the noncommutative Involutive Basis algorithm (Algorithm 12), together with an introduction to AlgLib, a set of ANSI C libraries providing data types and functions that serve as building blocks for the source code.

B.1 Methodology

A problem facing anyone wanting to implement mathematical ideas is the choice of language or system in which to do the implementation. The decision depends on the task at hand. If all that is required is a convenient environment for prototyping ideas, a symbolic computation system such as Maple [55], Mathematica [57] or MuPAD [49] may suffice. Such systems have a large collection of mathematical data types, functions and algorithms already present; tools that will not be available in a standard programming language. There is however always a price to pay for convenience. These common systems are all interpreted and use a proprietary programming syntax, making it it difficult to use other programs or libraries within a session. It also makes such systems less efficient than the execution of compiled programs.

The AlgLib libraries can be said to provide the best of both worlds, as they provide data types, functions and algorithms to allow programmers to more easily implement certain mathematical algorithms (including the algorithms described in this thesis) in the ANSI C programming language. For example, AlgLib contains the FMon [41] and FAlg [40]

libraries, respectively containing data types and functions to perform computations in the free monoid on a set of symbols and the free associative algebra on a set of symbols. Besides the benefit of the efficiency of compiled programs, the strict adherence to ANSI C makes programs written using the libraries highly portable.

B.1.1 MSSRC

AlgLib is supplied by MSSRC [46], a company whose Chief Scientist is Prof. Larry Lambe, an honorary professor at the University of Wales, Bangor. For an introduction to MSSRC, we quote the following passage from [42].

> Multidisciplinary Software Systems Research Corporation (MSSRC) was conceived as a company devoted to furthering the long-term effective use of mathematics and mathematical computation. MSSRC researches, develops, and markets advanced mathematical tools for engineers, scientists, researchers, educators, students and other serious users of mathematics. These tools are based on providing levels of power, productivity and convenience far greater than existing tools while maintaining mathematical rigor at all times. The company also provides computer education and training.
>
> MSSRC has several lines of ANSI C libraries for providing mathematical support for research and implementation of mathematical algorithms at various levels of complexity. No attempt is made to provide the user of these libraries with any form of Graphical User Interface (GUI). All components are compiled ANSI C functions which represent various mathematical operations from basic (adding, subtracting, multiplying polynomials, etc.) to advanced (operations in the free monoid on an arbitrary number of symbols and beyond). In order to use the libraries effectively, the user must be expert at ANSI C programming, e.g., in the style of Kernighan and Richie [38] and as such, they are not suited for the casual user. This does not imply in any way that excellent user interfaces for applications of the libraries cannot be supplied or are difficult to implement by well experienced programmers.
>
> The use of MSSRC's libraries has been reported in a number of places such as [43], [14], [16], [15] and elsewhere.

B.1.2 AlgLib

To give a taste of how AlgLib has been used to implement the algorithms considered in this thesis, consider one of the basic operations of these algorithms, the task of subtracting two polynomials to yield a third polynomial (an operation essential for computing an S-polynomial). In ordinary ANSI C, there is no data type for a polynomial, and certainly no function for subtracting two polynomials; AlgLib however does supply these data types and functions, both in the commutative and noncommutative cases. For example, the AlgLib data type for a noncommutative polynomial is an *FAlg*, and the AlgLib function for subtracting two such polynomials is the function *fAlgMinus*. It follows that we can write ANSI C code for subtracting two noncommutative polynomials, as illustrated below where we subtract the polynomial $2b^2 + ab + 4b$ from the polynomial $2 \times (b^2 + ba + 3a)$.

Source Code

```
# include <fralg.h>

int
main( argc, argv )
int argc;
char *argv[];
{
  // Define Variables
  FAlg p, q, r;
  QInteger two;

  // Set Monomial Ordering (DegLex)
  theOrdFun = fMonTLex;

  // Initialise Variables
  p = parseStrToFAlg("b^2 + b*a + 3*a");
  q = parseStrToFAlg("2*b^2 + a*b + 4*b");
  two = parseStrToQ("2");

  // Perform the calculation and display the result on screen
  r = fAlgMinus( fAlgScaTimes( two, p ), q );
  printf("2*(%s) - (%s) = %s\n", fAlgToStr( p ), fAlgToStr( q ), fAlgToStr( r ) );

  return EXIT_SUCCESS;
}
```

Program Output

```
ma6:mssrc-aux/thesis> fAlgMinusTest
2*(b^2 + b a + 3 a) - (2 b^2 + a b + 4 b) = 2 b a - a b -4 b + 6 a
ma6:mssrc-aux/thesis>
```

B.2 Listings

Our implementation of the noncommutative Involutive Basis algorithm is arranged as follows: *involutive.c* is the main program, dealing with all the input and output and calling the appropriate routines; the '*_functions*' files contain all the procedures and functions used by the program; and *README* describes how to use the program, including what format the input files should take and what the different options of the program are used for.

In more detail, *arithmetic_functions.c* contains functions for dividing a polynomial by its (coefficient) greatest common divisor and for converting user specified generators to ASCII generators (and vice-versa); *file_functions.c* contains all the functions needed to read and write polynomials and variables to and from disk; *fralg_functions.c* contains functions for monomial orderings, polynomial division and reduced Gröbner Bases computation; *list_functions.c* contains some extra functions needed to deal with displaying, sorting and manipulating lists; and *ncinv_functions.c* contains all the involutive routines, for example the Involutive Basis algorithm itself and associated functions for determining multiplicative variables and for performing autoreduction.

Contents

B.2.1 README

```
************************************************************
* HOW TO USE THE INVOLUTIVE PROGRAM - QUICK GUIDE *
************************************************************

NAME
        involutive  - Computes Noncommutative Involutive Bases for ideals.

SYNOPSIS
        involutive  [OPTION]... [FILE]...

DESCRIPTION
        Here are the options for the program.

        -a
            e.g. > involutive -d -a file.in
            Optimises the lexicographical ordering according to
            the frequency of the variables in the input basis
            (most frequent = lexicographically smallest).

        -c(n)
            e.g. > involutive -c2 file.in
            Chooses which involutive algorithm to use.
            n is a required number between 1 and 2.

            1: *DEFAULT * Gerdt's Algorithm
            2: Seiler's Algorithm

        -d
            e.g. > involutive -d file.in
            Allows the user to calculate a DegLex
            Involutive Basis for the basis in file.in.

        -e(n)
            e.g. > involutive -e2 -s2 file.in
            Allows the user to select the type of Overlap
            Division to use. n is a required number between
            1 and 5. Note: Must be used with either the
            -s1 or -s2 options.

            Left Overlap Division:

                 A        B            C           D
               ----- ------- --------- ------
            -----x    ----x       ----x       x---

            1: * DEFAULT * A, B, C (weak, Gr\"obner)
            2: A, B, C, Strong (strong if used with -m2)
            3: A, B, C, D (weak, Gr\"obner)
            4: A, B (weak, Gr\"obner)
            5: A (weak, Gr\"obner)
```

```
        Right Overlap Division:

            A            B          C           D
        -----      ------- --------- ------
          x-----       x---     x----     ---x

        1: * DEFAULT * A, B, C (weak, Gr\"obner)
        2: A, B, C, Strong (strong if used with -m2)
        3: A, B, C, D (weak, Gr\"obner)
        4: A, B (weak, Gr\"obner)
        5: A (weak, Gr\"obner)

    -f
        e.g. > involutive -f file.in
        Removes any fractions from the input basis.

    -l
        e.g. > involutive -l file.in
        Allows the user to calculate a Lex
        Involutive Basis for the basis in file.in.
        Warning: program may go into an infinite loop
        (Lex is not an admissible monomial ordering).

    -m(n)
        e.g. > involutive -m2 file.in
        Selects which method of deciding whether a monomial
        involutively divides another monomial is used.
        n is a required number between 1 and 2.

        1: * DEFAULT * 1st letters on left and right (thin divisor)
        2: All letters on left and right (thick divisor)

    -o(n)
        e.g. > involutive -o2 file.in
        Allows the user to select how the basis is sorted
        during the algorithm. n is a required number between
        1 and 3.

        1: * DEFAULT * DegRevLex Sorted
        2: No Sorting
        3: Sorting by Main Ordering

    -p
        e.g. > involutive -l -p file.in
        An interactive Ideal Membership Problem Solver.
        There are two ways the solver can be used:
        either a file containing a list of polynomials
        (e.g. x*y-z;
              x^2-z^2+y^2; ) can be given, or the
        polynomials can be input
        manually (e.g. x*y-z). The solver tests to see
        whether the Involutive Basis computed in the
        algorithm reduces the polynomials given to zero.
```

```
-r   * DEFAULT *
     e.g. > involutive -r file.in
     Allows the user to calculate a DegRevLex
     Involutive Basis for the basis in file.in.

-s(n)
     e.g. > involutive -s2 file.in
     Allows the user to select the type of Involutive
     Basis to calculate. n is a required number between
     1 and 5. Note: If an 'Overlap' Division is selected,
     the type of Overlap Division can be chosen with
     the -e(n) option.

     1: Left Overlap Division (local, cts, see -e option)
     2: Right Overlap Division (local, cts, see -e option)
     3: * DEFAULT * Left Division (global, cts, strong)
     4: Right Division (global, cts, strong)
     5: Empty Division (global, cts, strong)

-v(n)
     e.g. > involutive -v3 file.in
     Changes the amount of information given out by the
     program (i.e. the 'verbosity' of the program).
     n is a number between 0 and 9. Rough Guide:

     0: Silent (no output given).
     1: * DEFAULT *
     2: Returns Number of Reductions Carried Out,
        Prints Out Every Polynomial Found
     3: More Autoreduction Information,
        Prolongation Information
     4: More Details of Steps Taken in Algorithm
     5: More Global Division Information
     6: Step-by-Step Reduction, Overlap Information
     7: Shows Multiplicative Grids
     8: More Overlap Division Information
     9: All Other Information

-w
     e.g. > involutive -w file.in
     Allows the user to calculate an Involutive Basis
     for the basis in file.in using the Wreath
     Product Monomial Ordering.

-x
     e.g. > involutive -x file.in
     Ignores any prolongations of degree greater than or
     equal to 2d, where d is a value determined by the degree
     of the largest degree lead monomial in the current minimal basis.
     Warning: May not return a valid Involutive Basis
     (only a valid Gr\"obner Basis).
```

```
FILE FORMATS

        There is one file  format for the input basis:

        IDEALS:
        ---------------
        x; y; z;
        x*y - z;
        2*x + y*z + z;
        ---------------

        First line  = List of variables in order. In the
                      above, x; y; z; represents x > y > z.
        Remaining lines = Polynomial generators (which must be
                          terminated by semicolons).

OUTPUT

        As output, the program provides a reduced Gr\"obner Basis and
        an Involutive Basis for the input ideal (if it can calculate it).

        For the following, assume that our input basis was given as file.in.
        * If a DegRevLex Gr\"obner Basis is calculated, it is stored as file.drl.
        * If a DegLex Gr\"obner Basis is calculated, it is stored as file.deg.
        * If a Lex Gr\"obner Basis is calculated, it is stored as file.lex.
        * If a Wreath Product Gr\"obner Basis is calculated, it is stored as file.wp.

        The Involutive Basis is given as <Gr\"obner Basis>.inv.
        For example, if a DegLex Involutive Basis is calculated,
        it is stored as file.deg.inv.

        Note that the program has the ability to recognise the .in suffix and
        replace it with .drl, .deg, .lex or .wp as necessary.
        If your input file does not have a .in suffix then the program will
        simply append the appropriate suffix onto the end of the file name.
        For example, using the command
        > involutive FILE
        we obtain file.drl if FILE = file.in
        and obtain e.g. file.other.drl if FILE = file.other.
```

B.2.2 arithmetic_functions.h

```
/*
 * File: arithmetic_functions.h
 * Author: Gareth Evans
 * Last Modified: 29th September 2004
 */

// Initialise file definition
# ifndef ARITHMETIC_FUNCTIONS_HDR
# define ARITHMETIC_FUNCTIONS_HDR

```

```
// Include MSSRC Libraries
# include <fralg.h>

//
// Numerical Functions
//

// Returns the numerical value of a 3 letter word
ULong ASCIIVal( String );
// Returns the 3 letter word of a numerical value
String ASCIIStr( ULong );
// Returns the monomial corresponding to the 3 letter word of a numerical value
FMon ASCIIMon( ULong );

//
// QInteger Functions
//

// Calculate Alternative LCM of 2 QIntegers
QInteger AltLCMQInteger( QInteger, QInteger );

//
// FAlg Functions
//

// Divides the input FAlg by its common GCD
FAlg findGCD( FAlg );
// Returns maximal degree of lead term for the given FAlgList
ULong maxDegree( FAlgList );
// Returns the position of the smallest LM(g) in the given FAlgList
ULong fAlgListLowest( FAlgList );

# endif // ARITHMETIC_FUNCTIONS_HDR
```

B.2.3 arithmetic_functions.c

```
/*
 * File: arithmetic_functions.c
 * Author: Gareth Evans
 * Last Modified: 11th February 2005
 */

/*
 * ===================
 * Numerical Functions
 * ===================
 */

/*
 * Function Name: ASCIIVal
 *
 * Overview: Returns the numerical value of a 3 letter word
```

```
 *
 * Detail: Given a String containing 3 letters from the set
 * {A, B, ..., Z}, this function returns the numerical
 * value of the String according to the following rule:
 * AAA = 1, AAB = 2, ..., AAZ = 26, ABA = 27, ABB = 28,
 * ..., ABZ = 52, ACA = 53, ...
 *
 */
ULong
ASCIIVal( word )
String word;
{
  ULong back = 0;

  // Add on 17576*value of 1st letter (A = 0, B = 1, ...)
  back = back + 17576*( (ULong)( (int)word[0] - (int)'A' ) );
  // Add on 26*value of 2nd letter (A = 0, B = 1, ...)
  back = back + 26*( (ULong)( (int)word[1] - (int)'A' ) );
  // Add on the value of the 3rd letter (A = 1, B = 2, ...)
  back = back + (ULong)( (int)word[2] - (int)'A' + 1 );

  return back;
}

/*
 * Function Name: ASCIIStr
 *
 * Overview: Returns the 3 letter word of a numerical value
 *
 * Detail: Given a ULong, this function returns the
 * 3 letter String corresponding to the following rule:
 * 1 = AAA, 2 = AAB, ..., 26 = AAZ, 27 = ABA, 28 = ABB,
 * ..., 52 = ABZ, 53 = ACA, ...
 *
 */
String
ASCIIStr( number )
ULong number;
{
  String back = strNew();
  int i = 0, j = 0, k;

  // Take away multiples of 26^2 to get the first letter
  while( number > 17576 )
  {
    i++;
    number = number - 17576;
  }

  // Take away multiples of 26 to get the second letter
  while( number > 26 )
  {
    j++;
```

```
    number = number - 26;
  }

  // We are now left with the third letter
  k = (int) number - 1;

  // Convert the numbers to a String
  sprintf( back, "%c%c%c", (char)( (int)'A' + i ),
                           (char)( (int)'A' + j ),
                           (char)( (int)'A' + k ) );

  // Return the three letters
  return back;
}

/*
 * Function Name: ASCIIStr
 *
 * Overview: Returns the monomial corresponding to the
 * 3 letter word of a numerical value
 *
 * Detail: Given a ULong, this function returns the
 * monomial corresponding to the following rule:
 * 1 = AAA, 2 = AAB, ..., 26 = AAZ, 27 = ABA, 28 = ABB,
 * ..., 52 = ABZ, 53 = ACA, ...
 *
 */
FMon
ASCIIMon( number )
ULong number;
{
  // Obtain the String corresponding to the input
  // number and change it to an FMon
  return parseStrToFMon( ASCIIStr( number ) );
}

/*
 * ==================
 * QInteger Functions
 * ==================
 */

/*
 * Function Name: AltLCMQInteger
 *
 * Overview: Calculates an 'alternative' LCM of 2 QIntegers
 *
 * Detail: Given two QIntegers a = an/ad and b = bn/bd,
 * this function calculates the LCM given
 * by alt_lcm(a, b) = (a*b)/(alt_gcd(a, b))
 * = (an*bn*ad*bd)/(ad*bd*gcd(an, bn)*gcd(ad, bd))
 * = (an*bn)/(gcd(an, bn)*gcd(ad, bd)).
 *
```

```
 */
QInteger
AltLCMQInteger( a, b )
QInteger a, b;
{
  Integer an = a −> num,
          ad = a −> den,
          bn = b −> num,
          bd = b −> den;

  return qDivide( zToQ( zTimes( an, bn ) ),
                  zToQ( zTimes( zGcd( an, bn ), zGcd( ad, bd ) ) ) );
}

/*
 * ==============
 * FAlg Functions
 * ==============
 */

/*
 * Function Name: findGCD
 *
 * Overview: Divides the input FAlg by its common GCD
 *
 * Detail: Given an FAlg, this function divides the
 * polynomial by its common GCD so that the output
 * polynomial g cannot be written as g = cg', where
 * g' is a polynomial and c is an integer, c > 1.
 *
 */
FAlg
findGCD( input )
FAlg input;
{
  FAlg output = input, process = input;
  QInteger coef;
  Integer GCD = zOne, numerator, denominator;
  Bool first = 0, allNeg = qLess( fAlgLeadCoef( input ), qZero() );

  if( (ULong) fAlgNumTerms( input ) == 1 ) // If poly has just 1 term
  {
    // Return that term with a unit coefficient
    return fAlgMonom( qOne(), fAlgLeadMonom( input ) );
  }
  else // Poly has more than 1 term
  {
    while( process ) // Go through each term
    {
      coef = fAlgLeadCoef( process ); // Read the lead coefficient
      numerator = coef −> num; // Break the coefficient down
      denominator = coef −> den; // into a numerator and a denominator
      process = fAlgReductum( process ); // Get ready to look at the next term
```

```

      if( zIsOne( denominator ) != (Bool) 1 ) // If we encounter a fraction
      {
        return input; // We cannot divide through by a GCD so just return the input
      }
      else // The coefficient was an integer
      {
        if( first == 0 ) // If this is the first term
        {
          first = (Bool) 1;
          GCD = numerator; // Set the GCD to be the current numerator
        }
        else // Recursively calculate the GCD
          GCD = zGcd( GCD, numerator );
      }
    }

    if( zLess( GCD, zZero ) == (Bool) 1 ) // If the GCD is negative
      GCD = zNegate( GCD ); // Negate the GCD
    if( zLess( zOne, GCD ) == (Bool) 1 ) // If the GCD is > 1
      output = fAlgZScaDiv( output, GCD ); // Divide the poly by the GCD
  }

  if( allNeg == (Bool) 1 ) // If the original coefficient was negative
    return fAlgZScaTimes( zMinusOne, output ); // Return the negated polynomial
  else
    return output;
}

/*
 * Function Name: maxDegree
 *
 * Overview: Returns maximal degree of lead term for the given FAlgList
 *
 * Detail: Given an FAlgList, this function calculates the degree
 * of the lead term for each element of the list and returns
 * the largest value found.
 *
 */
ULong
maxDegree( input )
FAlgList input;
{
  ULong test, output = 0;

  while( input ) // For each polynomial in the list
  {
    // Calculate the degree of the lead monomial
    test = fMonLength( fAlgLeadMonom( input -> first ) );
    if( test > output ) output = test;
    input = input -> rest; // Advance the list
  }

```

```
  // Return the maximal value
  return output;
}

/*
 * Function Name: fAlgListLowest
 *
 * Overview: Returns the position of the smallest LM(g) in the given FAlgList
 *
 * Detail: Given an FAlgList, this function looks at all the leading
 * monomials of the elements in the list and returns the position of
 * the smallest lead monomial with respect to the monomial ordering
 * currently being used.
 *
 */
ULong
fAlgListLowest( input )
FAlgList input;
{
  ULong output = 0, i, len = fAlgListLength( input );
  FMon next, lowest;

  if( input ) // Assume the 1st lead monomial is the smallest to begin with
  {
    lowest = fAlgLeadMonom( input -> first );
    output = 1;
  }
  for( i = 1; i < len; i++ ) // For the remaining polynomials
  {
    input = input -> rest;
    // Extract the next lead monomial
    next = fAlgLeadMonom( input -> first );

    // If this lead monomial is smaller than the current smallest
    if( theOrdFun( next, lowest ) == (Bool) 1 )
    {
      // Make this lead monomial the smallest
      output = i+1;
      lowest = fAlgScaTimes( qOne(), next );
    }
  }

  // Return position of smallest lead monomial
  return output;
}

/*
 * ===========
 * End of File
 * ===========
 */
```

B.2.4 file_functions.h

```
/*
 * File: file_functions.h
 * Author: Gareth Evans
 * Last Modified: 14th July 2004
 */

// Initialise file definition
# ifndef FILE_FUNCTIONS_HDR
# define FILE_FUNCTIONS_HDR

// Include MSSRC Libraries
# include <fralg.h>

// MAXLINE denotes the length of the longest allowable line in a file
# define MAXLINE 5000

//
// Low Level File Handling Functions
//

// Read a line from a file; return length
int getLine( FILE *, char[], int );
// Pick an integer from a list such as "2, 5, 6,"
int intFromStr( char[], int, int * );
// Pick a variable from a list such as "a; b; c;"
String variableFromStr( char[], int, int * );
// Pick an FMon from a list such as "a; b; c;"
FMon fMonFromStr( char[], int, int * );
// Pick an FAlg from a string such as "x*y - z;"
FAlg fAlgFromStr( char[], int, int * );

//
// High Level File Reading Functions
//

// Routine to read an FMonList from the first line of a file
FMonList fMonListFromFile( FILE * );
// Routine to read an FAlgList from a file
FAlgList fAlgListFromFile( FILE * );

//
// High Level File Writing Functions
//

// Writes an FMon (in parse format) followed by a semicolon to a file
void fMonToFile( FILE *, FMon );
// Writes an FMonList to a file on a single line
void fMonListToFile( FILE *, FMonList );

//
// File Name Modification Functions
```

```
//

// Appends ".drl" onto a string (except in special case "*.in")
String appendDotDegRevLex( char[] );
// Appends ".deg" onto a string (except in special case "*.in")
String appendDotDegLex( char[] );
// Appends ".lex" onto a string (except in special case "*.in")
String appendDotLex( char[] );
// Appends ".wp" onto a string (except in special case "*.in")
String appendDotWP( char[] );
// Calculates the length of an input string
int filenameLength( char[] );

# endif // FILE_FUNCTIONS_HDR
```

B.2.5 file_functions.c

```
/*
 * File: file_functions.c
 * Author: Gareth Evans
 * Last Modified: 16th August 2004
 */

/*
 * ===================================
 * Low Level File Handling Functions
 * (Used in the high level functions)
 * ===================================
 */

/*
 * Function Name: getLine
 *
 * Overview: Read a line from a file; return length
 *
 * Detail: Given a file _infil_, we read the first line
 * of the file, placing the contents into the string _s_.
 * The third parameter _lim_ determines the maximum length
 * of any line to be returned (when we call the function
 * this is usually MAXLINE); the returned integer tells
 * us the length of the line we have just read.
 *
 * Known Issues: The length of a line is sometimes returned
 * incorrectly when a file saved in Windows is used
 * on a UNIX machine. Resave your file in UNIX.
 */
int
getLine( infil, s, lim )
FILE *infil;
char s[];
int lim;
{
```

```
  int c, i;

  /*
   * Place characters in _s_ as long as (1) we do not exceed _lim_ number of
   * characters; (2) the end of the file is not encountered; (3) the end of the
   * line is not encountered.
   */
  for( i = 0; ( i < lim-1 ) && ( ( c = fgetc(infil) ) != -1 ) && ( c != (int)'\n' ); i++ )
  {
    s[i] = (char)c;
  }
  if( c == (int)'\n' ) // if the for loop was terminated due to reaching end of line
  {
    s[i] = (char)c; // add the newline character to our string
    i++;
  }
  s[i] = '\0'; // '\0' is the null character

  return i-1; // The -1 is used to compensate for the null character
}

/*
 * Function Name: intFromStr
 *
 * Overview: Pick an integer from a list such as "2, 5, 6,"
 *
 * Detail: Starting from position _j_ in a string _s_,
 * read in an integer and return it. Note that the integer
 * in the string must be terminated with a comma and that
 * the sign of the integer is taken into account.
 * Once the integer has been read, place the position we
 * have reached in the string in the variable _pk_.
 */
int
intFromStr( s, j, pk )
char s[];
int j, *pk;
{
  char c;
  int n = 0, sign = 1, k = j;
  c = s[k];

  // Traverse through any empty space
  while( c == ' ' )
  {
    k++;
    c = s[k];
  }

  // If a sign is present, process it
  if( c == '+' )
  {
    k++;
```

```
    c = s[k];
  }
  else if( c == '-' )
  {
    sign = -1;
    k++;
    c = s[k];
  }

  // Until a comma is encountered (signalling the
  // end of the integer)
  while( c != ',' )
  {
    if( ( c >= '0' ) && ( c <= '9' ) )
    {
      n = 10*n + (int)(c - '0'); // the "- '0'" is needed to get the correct integer
    }
    else
    {
      printf("Error: Incorrect Input in File (%c is not a number).\n", c);
      exit( EXIT_FAILURE );
    }
    k++;
    c = s[k];
  }
  *pk = k+1; // return the finishing position

  /*
   * Note: In this function we return *pk = k+1 and not *pk = k as
   * in subsequent functions because this function has a slightly
   * different structure due to having to deal with the + and -
   * characters at the beginning of the string.
   */

  return sign*n; // return the integer
}

/*
 * Function Name: variableFromStr
 *
 * Overview: Pick a variable from a list such as "a; b; c;"
 *
 * Detail: Starting from position _j_ in a string _s_,
 * read in a String and return it. Note that the String
 * in the string must be terminated with a semicolon.
 * Once the String has been read, place the position we
 * have reached in the variable _pk_.
 */
String
variableFromStr( s, j, pk )
char s[];
int j, *pk;
{
```

```
  char c = '␣';
  int i = 0, k = j;
  String back = strNew(), concat;

  sprintf( back, "" ); // Initialise back

  // Until a semicolon is encountered
  while( c != ';' )
  {
    c = s[k]; // Pick a character from the string

    // If a semicolon was encountered
    if( c == ';' )
    {
      concat = strNew();
      sprintf( concat, "%c", '\0' );
      back = strConcat( back, concat ); // Finish with the null character
    }
    else if( c != '␣' )
    {
      concat = strNew();
      sprintf( concat, "%c", c );
      // Transfer character to output String
      if( i == 0 ) back = strCopy( concat );
      else back = strConcat( back, concat );
      i++;
    }
    k++;
  }
  *pk = k; // Place finish position in the variable _pk_

  return back; // Return the String
}

/*
 * Function Name: fMonFromStr
 *
 * Overview: Pick an FMon from a list such as "a; b; c;"
 *
 * Detail: Starting from position _j_ in a string _s_,
 * read in an FMon and return it. Note that the FMon
 * in the string must be terminated with a semicolon.
 * Once the FMon has been read, place the position we
 * have reached in the string in the variable _pk_.
 */
FMon
fMonFromStr( s, j, pk )
char s[];
int j, *pk;
{
  char c = '␣', a[MAXLINE];
  int i = 0, k = j;
  FMon back;
```

```

  // Until a semicolon is encountered
  while( c != ';' )
  {
    c = s[k]; // Pick a character from the string

    // If we have found a semicolon
    if( c == ';' )
    {
      a[i] = '\0'; // Finish the string with the null character
    }
    else
    {
      a[i] = c; // Continue to process...
      i++;
    }
    k++;
  }
  *pk = k; // Place the finish position in the variable _pk_

  back = parseStrToFMon( a ); // Convert the string to an FMon
  return back; // Return the FMon
}

/*
 * Function Name: fAlgFromStr
 *
 * Overview: Pick an FAlg from a string such as "x*y - z;"
 *
 * Detail: Starting from position _j_ in a string _s_,
 * read in an FAlg and return it. Note that the FAlg
 * in the string must be terminated with a semicolon.
 * Once the FAlg has been read, place the position we
 * have reached in the string in the variable _pk_.
 */
FAlg
fAlgFromStr( s, j, pk )
char s[];
int j, *pk;
{
  char c = ' ', a[MAXLINE];
  int i = 0, k = j;
  FAlg back;

  // Until a semicolon is encountered
  while( c != ';' )
  {
    c = s[k]; // Read a character from the string

    // If a semicolon is encountered
    if( c == ';' )
    {
      a[i] = '\0'; // Finish with the null character
```

```
    }
    else
    {
      a[i] = c; // Continue to process...
      i++;
    }
    k++;
  }
  *pk = k; // Place the finish position in the variable _pk_

  back = parseStrToFAlg( a ); // Convert the string to an FAlg
  return back; // Return the FAlg
}

/*
 * ==================================
 * High Level File Reading Functions
 * ==================================
 */

/*
 * Function Name: fMonListFromFile
 *
 * Overview: Routine to read an FMonList from the first line of a file
 *
 * Detail: Given an input file, this function
 * reads the first line of the file and returns
 * the semicolon separated FMonList found on that line.
 * For example, if the input is a list such as a; b; A; B;
 * then the output is the FMonList (a, b, A, B).
 */
FMonList
fMonListFromFile( infil )
FILE *infil;
{
  FMon w;
  FMonList words = fMonListNul;
  char s[MAXLINE];
  int j = 0, k = 0, len = 0;

  // Get the first line of the file and its length
  len = getLine( infil, s, MAXLINE );

  // While there are more FMons to be found
  while( j < len )
  {
    w = fMonFromStr( s, j, &k ); // Obtain an FMon
    j = k; // Set the next starting position
    words = fMonListPush( w, words ); // Construct the list
  }

  // Return the list - note that we must reverse the list
  // because it has been read in reverse order.
```

```
  return fMonListFXRev( words );
}

/*
 * Function Name: fAlgListFromFile
 *
 * Overview: Routine to read an FAlgList from a file
 *
 * Detail: Given an input file, this function
 * takes each line of the file in turn, pushing one FAlg from
 * each line onto an FAlgList. This process is
 * continued until there are no more lines in the file
 * to process. For example, if the input is a list such as
 *
 * 2*x - 4*y;
 * 5*x*y;
 * 4 + 5*x + 60*y;
 *
 * then the output is the FAlgList
 * (2x-4y, 5xy, 4+5x+60y).
 */
FAlgList
fAlgListFromFile( infil )
FILE *infil;
{
  FAlg entry;
  FAlgList back = fAlgListNul;
  char s[MAXLINE];
  int j = 0, k = 0, len;

  // Get the first line of the file
  len = getLine( infil, s, MAXLINE );

  // While there are still lines to process
  while( len > 0 )
  {
    entry = fAlgFromStr( s, j, &k ); // Obtain an FAlg from a line
    back = fAlgListPush( entry, back ); // Push the FAlg onto the list
    len = getLine( infil, s, MAXLINE ); // Get a new line
  }

  // Return the list - note that we must reverse the list
  // because it has been read in reverse order.
  return fAlgListFXRev( back );
}

/*
 * ==================================
 * High Level File Writing Functions
 * ==================================
 */

/*
```

```
 * Function Name: fMonToFile
 *
 * Overview: Writes an FMon (in parse format) followed by a semicolon to a file
 *
 * Detail: Given an input file and an FMon, this function
 * writes the FMon to file in parse format followed by a semicolon.
 */
void
fMonToFile( infil, w )
FILE *infil;
FMon w;
{
  FMon wM;
  ULong length;

  // If the FMon is non-empty
  if ( fMonEqual( w, fMonOne() ) != (Bool) 1 )
  {
    // While there are letters left in the FMon
    while ( w )
    {
      wM = fMonLeadPowFac( w ); // Obtain a factor
      fprintf( infil, "%s", fMonToStr( wM ) ); // Write the factor to file
      length = fMonLength( wM );
      w = fMonSuffix( w, fMonLength( w ) - length );
      if ( fMonEqual( w, fMonOne() ) != (Bool) 1 )
      {
        // In parse format, to separate variables we use an asterisk
        fprintf( infil, "*" );
      }
    }
    fprintf( infil, ";" ); // At the end write a semicolon to file
  }
  else // Just write a semicolon to file
  {
    fprintf( infil, ";" );
  }
}

/*
 * Function Name: fMonListToFile
 *
 * Overview: Writes an FMonList to a file on a single line
 *
 * Detail: Given an input file and an FMonList, this function
 * writes the list to file as l1; l2; l3; ...
 */
void
fMonListToFile( infil, L )
FILE *infil;
FMonList L;
{
  ULong i, length = fMonListLength( L );
```

```

  // For each element of the list
  for( i = 1; i <= length; i++ )
  {
    // Write an FMon to file
    fMonToFile( infil, L -> first );

    // If there are more FMons left to look at
    if( i < length )
    {
      fprintf( infil, " " ); // Provide a space between elements
    }
    else // else terminate the line
    {
     fprintf( infil, "\n" );
    }
    L = L -> rest;
  }
}

/*
 * ================================
 * File Name Modification Functions
 * ================================
 */

/*
 * Function Name: appendDotDegRevLex
 *
 * Overview: Appends ".drl" onto a string (except in special case "*.in")
 *
 * Detail: Given an input character array, this function
 * appends the String ".drl" onto the end of the character array.
 * In the special case that the input ends with ".in", the function
 * replaces the ".in" with ".drl".
 */
String
appendDotDegRevLex( input )
char input[];
{
  int length = (int) strlen( input );
  String back = strNew();

  // First check for .in at the end of the file name
  if ( input[length-1] == 'n' & input[length-2] == 'i' & input[length-3] == '.' )
  {
    input[length-2] = 'd';
    input[length-1] = 'r';
    sprintf( back, "%s%s", input, "l" );
  }
  else // Just append with ".drl"
  {
    sprintf( back, "%s%s", input, ".drl" );
```

```
  }

  return back;
}

/*
 * Function Name: appendDotDegLex
 *
 * Overview: Appends ".deg" onto a string (except in special case "*.in")
 *
 * Detail: Given an input character array, this function
 * appends the String ".deg" onto the end of the character array.
 * In the special case that the input ends with ".in", the function
 * replaces the ".in" with ".deg".
 */
String
appendDotDegLex( input )
char input[];
{
  int length = (int) strlen( input );
  String back = strNew();

  // First check for .in at the end of the file name
  if ( input[length-1] == 'n' & input[length-2] == 'i' & input[length-3] == '.' )
  {
    input[length-2] = 'd';
    input[length-1] = 'e';
    sprintf( back, "%s%s", input, "g" );
  }
  else // Just append with ".deg"
  {
    sprintf( back, "%s%s", input, ".deg" );
  }

  return back;
}

/*
 * Function Name: appendDotLex
 *
 * Overview: Appends ".lex" onto a string (except in special case "*.in")
 *
 * Detail: Given an input character array, this function
 * appends the String ".lex" onto the end of the character array.
 * In the special case that the input ends with ".in", the function
 * replaces the ".in" with ".lex".
 */
String
appendDotLex( input )
char input[];
{
  int length = (int) strlen( input );
  String back = strNew();
```

```

  // First check for .in at the end of the file name
  if ( input[length-1] == 'n' & input[length-2] == 'i' & input[length-3] == '.' )
  {
    input[length-2] = 'l';
    input[length-1] = 'e';
    sprintf( back, "%s%s", input, "x" );
  }
  else // Just append with ".lex"
  {
    sprintf( back, "%s%s", input, ".lex" );
  }

  return back;
}

/*
 * Function Name: appendDotWP
 *
 * Overview: Appends ".wp" onto a string (except in special case "*.in")
 *
 * Detail: Given an input character array, this function
 * appends the String ".wp" onto the end of the character array.
 * In the special case that the input ends with ".in", the function
 * replaces the ".in" with ".wp".
 */
String
appendDotWP( input )
char input[];
{
  int length = (int) strlen( input );
  String back = strNew();

  // First check for .in at the end of the file name
  if ( input[length-1] == 'n' & input[length-2] == 'i' & input[length-3] == '.' )
  {
    input[length-2] = 'w';
    input[length-1] = 'p';
    sprintf( back, "%s", input );
  }
  else // Just append with ".wp"
  {
    sprintf( back, "%s%s", input, ".wp" );
  }

  return back;
}

/*
 * Function Name: filenameLength
 *
 * Overview: Calculates the length of an input string
 *
```

```
 * Detail: Given an input character array, this function
 * finds the length of that character array
 */
int
filenameLength( s )
char s[];
{
  int i = 0;

  while( s[i] != '\0' ) i++;

  return i;
}

/*
 * ===========
 * End of File
 * ===========
 */
```

B.2.6 fralg_functions.h

```
/*
 * File: fralg_functions.h
 * Author: Gareth Evans
 * Last Modified: 10th August 2005
 */

// Initialise file definition
# ifndef FRALG_FUNCTIONS_HDR
# define FRALG_FUNCTIONS_HDR

// Include MSSRC Libraries
# include <fralg.h>

// Include System Libraries
# include <limits.h>

// Include *_functions Libraries
# include "list_functions.h"
# include "arithmetic_functions.h"

//
// External Variables Required
//

extern ULong nRed; // Stores how many reductions have been performed
extern int nOfGenerators, // Holds the number of generators
           pl; // Holds the "Print Level"

//
// Functions Defined in fralg_functions.c
```

```
//

//
// Ordering Functions
//

// Returns 1 if 1st arg <_{Lex} 2nd arg
Bool fMonLex( FMon, FMon );
// Returns 1 if 1st arg <_{InvLex} 2nd arg
Bool fMonInvLex( FMon, FMon );
// Returns 1 if 1st arg <_{DegRevLex} 2nd arg
Bool fMonDegRevLex( FMon, FMon );
// Returns 1 if 1st arg <_{WreathProduct} 2nd arg
Bool fMonWreathProd( FMon, FMon );

//
// Alphabet Manipulation Functions
//

// Substitutes ASCII generators for original generators in a list of polynomials
FAlgList preProcess( FAlgList, FMonList );
// Substitutes original generators for ASCII generators in a given polynomial
String postProcess( FAlg, FMonList );
// As above but gives back its output in parse format
String postProcessParse( FAlg, FMonList );
// Adjusts the original generator order (1st arg) according to frequency of generators in 2nd arg
FMonList alphabetOptimise( FMonList, FAlgList );

//
// Polynomial Manipulation Functions
//

// Returns all possible ways that 2nd arg divides 1st arg; 3rd arg = is division possible?
FMonPairList fMonDiv( FMon, FMon, Short * );
// Returns the first way that 2nd arg divides 1st arg; 3rd arg = is division possible?
FMonPairList fMonDivFirst( FMon, FMon, Short * );
// Finds all possible overlaps of 2 FMons
FMonPairList fMonOverlaps( FMon, FMon );
// Returns the degree-based initial of a polynomial
FAlg degInitial( FAlg );
// Reverses a monomial
FMon fMonReverse( FMon );

//
// Groebner Basis Functions
//

// Returns the normal form of a polynomial w.r.t. a list of polynomials
FAlg polyReduce( FAlg, FAlgList );
// Minimises a given Groebner Basis
FAlgList minimalGB( FAlgList );
// Reduces each member of a Groebner Basis w.r.t. all other members
FAlgList reducedGB( FAlgList );
```

```
// Tests whether a given FAlg reduces to 0 using the given FAlgList
Bool idealMembershipProblem( FAlg, FAlgList );

# endif // FRALG_FUNCTIONS_HDR
```

B.2.7 fralg_functions.c

```
/*
 * File: fralg_functions.c
 * Author: Gareth Evans
 * Last Modified: 10th August 2005
 */

/*
 * ========================================
 * Global Variables for fralg_functions.c
 * ========================================
 */

static int bigVar = 1; // Keeps track of iteration depth in WreathProd

/*
 * ===================
 * Ordering Functions
 * ===================
 */

/*
 * Function Name: fMonLex
 *
 * Overview: Returns 1 if 1st arg <_{Lex} 2nd arg
 *
 * Detail: Given two FMons x and y, this function
 * compares the two monomials using the lexicographic
 * ordering, returning 1 if x < y and 0 if x >= y.
 *
 * Description of the Lex ordering:
 *
 * x < y iff (working left-to-right) the first (say ith)
 * letter on which x and y differ is
 * such that x_i < y_i in the ordering of the variables.
 *
 * External Variables Required: int pl;
 *
 * Note: This code is based on L. Lambe's "fMonTLex" code.
 */
Bool
fMonLex( x, y )
FMon x, y;
{
  ULong lenx, leny, min, count = 1;
  int j;
```

```
Bool back;

if( pl > 8 ) printf("Entered to compare x = %s with y = %s... \n", fMonToStr( x ), fMonToStr( y ) );
if( x == (FMon) NULL ) // If x is empty we only have to check that y is non−empty
{
  if( pl > 8 ) printf("x is NULL so testing if y is NULL...\n");
  return (Bool) ( y != (FMon) NULL );
}
else if( y == (FMon) NULL ) // If y is empty x cannot be less than it so just return 0
{
  if( pl > 8 ) printf("y is NULL so returning 0...\n");
  return (Bool) 0;
}
else // Both non−empty
{
  lenx = fMonLength( x );
  leny = fMonLength( y );

  if( lenx < leny ) // x has minimum length
  {
    min = lenx;
    back = (Bool) 1; // If limit reached we know x < y so return 1
  }
  else // y has minimum length
  {
    min = leny;
    back = (Bool) 0; // if limit reached we know x >= y so return 0
  }

  while( count <= min ) // For each generator
  {
    if( pl > 8 )
    {
      printf("Comparing %s with %s\n", fMonLeadVar( fMonSubWordLen( x, count, 1 ) ),
                                       fMonLeadVar( fMonSubWordLen( y, count, 1 ) ) );
    }
    // Compare generators
    if( ( j = strcmp( fMonLeadVar( fMonSubWordLen( x, count, 1 ) ),
                      fMonLeadVar( fMonSubWordLen( y, count, 1 ) ) ) ) < 0 )
    {
      if( pl > 8 ) printf("x is less than y... \n");
      return (Bool) 1;
    }
    else if( j > 0 )
    {
      if( pl > 8 ) printf("y is less than x... \n");
      return (Bool) 0;
    }
    count++;
  }
}

// Limit now reached; return previously agreed solution
```

```
  if( pl > 8 ) printf("Returning␣%i...␣\n", (int) back);
  return back;
}

/*
 * Function Name: fMonInvLex
 *
 * Overview: Returns 1 if 1st arg <_{InvLex} 2nd arg
 *
 * Detail: Given two FMons x and y, this function
 * compares the two monomials using the inverse lexicographic
 * ordering, returning 1 if x < y and 0 if x >= y.
 *
 * Description of the InvLex ordering:
 *
 * x < y iff (working right-to-left) the first (say ith)
 * letter on which x and y differ is
 * such that x_i < y_i in the ordering of the variables.
 *
 * External Variables Required: int pl;
 *
 * Note: This code is based on L. Lambe's "fMonTLex" code.
 */
Bool
fMonInvLex( x, y )
FMon x, y;
{
  ULong lenx, leny, min, count = 0;
  int j;
  Bool back;

  if( pl > 8 ) printf("Entered␣to␣compare␣x␣=␣%s␣with␣y␣=␣%s...␣\n", fMonToStr( x ), fMonToStr( y ) );
  if( x == (FMon) NULL ) // If x is empty we only have to check that y is non-empty
  {
    if( pl > 8 ) printf("x␣is␣NULL␣so␣testing␣if␣y␣is␣NULL...\n");
    return (Bool) ( y != (FMon) NULL );
  }
  else if( y == (FMon) NULL ) // If y is empty x cannot be less than it so just return 0
  {
    if( pl > 8 ) printf("y␣is␣NULL␣so␣returning␣0...\n");
    return (Bool) 0;
  }
  else // Both non-empty
  {
    lenx = fMonLength( x );
    leny = fMonLength( y );

    if( lenx < leny ) // x has minimum length
    {
      min = lenx;
      back = (Bool) 1; // If limit reached we know x < y so return 1
    }
    else // y has minimum length
```

```
    {
      min = leny;
      back = (Bool) 0; // if limit reached we know x >= y so return 0
    }

    while( count < min ) // For each generator
    {
      if( pl > 8 )
      {
        printf("Comparing %s with %s\n", fMonLeadVar( fMonSubWordLen( x, lenx-count, 1 ) ),
                                         fMonLeadVar( fMonSubWordLen( y, leny-count, 1 ) ) );
      }
      // Compare generators _in reverse_
      if( ( j = strcmp( fMonLeadVar( fMonSubWordLen( x, lenx-count, 1 ) ),
                        fMonLeadVar( fMonSubWordLen( y, leny-count, 1 ) ) ) ) < 0 )
      {
        if( pl > 8 ) printf("x is less than y... \n");
        return (Bool) 1;
      }
      else if( j > 0 )
      {
        if( pl > 8 ) printf("y is less than x... \n");
        return (Bool) 0;
      }
      count++;
    }
  }

  // Limit now reached; return previously agreed solution
  if( pl > 8 ) printf("Returning %i... \n", (int) back);
  return back;
}

/*
 * Function Name: fMonDegRevLex
 *
 * Overview: Returns 1 if 1st arg <_{DegRevLex} 2nd arg
 *
 * Detail: Given two FMons x and y, this function
 * compares the two monomials using the degree reverse lexicographic
 * ordering, returning 1 if x < y and 0 if x >= y.
 *
 * Description of the DegRevLex ordering:
 *
 * x < y iff deg(x) < deg(y) or deg(x) = deg(y)
 * and x <_{RevLex} y, that is, working right to left,
 * the first (say ith) letter on which x and y differ is
 * such that x_i > y_i in the ordering of the variables.
 *
 * External Variables Required: int pl;
 *
 * Note: This code is based on L. Lambe's "fMonTLex" code.
 */
```

```
Bool
fMonDegRevLex( x, y )
FMon x, y;
{
  ULong lenx, leny, count;
  int j;

  if( pl > 8 ) printf("Entered to compare x = %s with y = %s... \n", fMonToStr( x ), fMonToStr( y ) );

  if( x == (FMon) NULL ) // If x is empty we only have to check that y is non-empty
  {
    if( pl > 8 ) printf("x is NULL so testing if y is NULL...\n");
    return (Bool) ( y != (FMon) NULL );
  }
  else if( y == (FMon) NULL ) // If y is empty x cannot be less than it so just return 0
  {
    if( pl > 8 ) printf("y is NULL so returning 0...\n");
    return (Bool) 0;
  }
  else // Both non-empty
  {
    lenx = fMonLength( x );
    leny = fMonLength( y );

    // In DegRevLex, compare the degrees first...
    if( lenx < leny )
    {
      if( pl > 8 ) printf("x is less than y... \n");
      return (Bool) 1;
    }
    else if( leny < lenx )
    {
      if( pl > 8 ) printf("y is less than x... \n");
      return (Bool) 0;
    }
    else // The degrees are the same, now use RevLex...
    {
      count = lenx; // lenx is arbitrary (because lenx = leny)

      while( count > 0 ) // Work in _reverse_
      {
        if( pl > 8 )
        {
          printf("Comparing %s with %s\n", fMonLeadVar( fMonSubWordLen( x, count, 1 ) ),
                                            fMonLeadVar( fMonSubWordLen( y, count, 1 ) ) );
        }
        if( ( j = strcmp( fMonLeadVar( fMonSubWordLen( x, count, 1 ) ),
                          fMonLeadVar( fMonSubWordLen( y, count, 1 ) ) ) ) > 0 )
        {
          if( pl > 8 ) printf("x is less than y... \n");
          return (Bool) 1;
        }
        else if( j < 0 )
```

```
          {
            if( pl > 8 ) printf("y is less than x... \n");
            return (Bool) 0;
          }
          count--;
        }
      }
  }

  // No differences found so monomials must be the same
  if( pl > 8 ) printf("Same, returning 0... \n");
  return (Bool) 0;
}

/*
 * Function Name: fMonWreathProd
 *
 * Overview: Returns 1 if 1st arg <_{WreathProduct} 2nd arg
 *
 * Detail: Given two FMons x and y, this function
 * compares the two monomials using the wreath product
 * ordering, returning 1 if x < y and 0 if x >= y.
 * This function is recursive.
 *
 * Description of the Wreath Product Ordering:
 *
 * Let the alphabet have a total order (e.g. a < b < ...)
 * Count the number of occurrences of the highest weighted letter (e.g. z),
 * the string with the most is bigger.
 * If both strings have the same number of those letters, they can
 * be written uniquely:
 * s1 = x0 z x1 z x2 ... z xn
 * s2 = y0 z y1 z y2 ... z yn
 *
 * Then s1 < s2 if
 * x0 < y0 or
 * x0 = y0 and x1 < y1, etc.
 * (< = wreath product ordering 'on y'; iterate as needed)
 *
 * Examples:
 * a^100 < aba^2 because 1 < b
 * aba^2 < a^2ba because b = b and a < a^2
 * a^2ba < b^2a because b < b^2
 * b^2a < bab because b^2 = b^2 and 1 < a (s1 = 1b1ba and s2 = 1bab1)
 * bab < ab^2 because b^2 = b^2 and 1 < a (s1 = 1bab1 and s2 = ab1b1)
 *
 * External Variables Required: int pl, nOfGenerators;
 * Global Variables Used: int bigVar;
 *
 * Note: This code is based on L. Lambe's "fMonTLex" code.
 */
Bool
fMonWreathProd( x, y )
```

```
FMon x, y;
{
  FMonList xList = fMonListNul, yList = fMonListNul;
  FMon xPad = fMonOne(), yPad = fMonOne(), xLetter, yLetter, bigMon;
  ULong xCount = 0, yCount = 0, i = 0;

  /*
   * Note: the global variable 'bigVar' is used to keep
   * track of the iteration depth. The algorithm is designed
   * so that the value of bigVar is always returned to its
   * original value (which is usually 1)
   */

  if( pl > 8 ) printf("Entered fMonWreathProd (%i) to compare x = %s with y = %s... \n",
                      bigVar, fMonToStr( x ), fMonToStr( y ) );

  // Fail safe check - cannot have more iterations than generators;
  // value 1 chosen by convention (in the case of equality)
  if( !( nOfGenerators-bigVar >= 0 ) ) return (Bool) 1;

  // Deal with special cases first
  if( x == (FMon) NULL ) // If x is empty we only have to check that y is non-empty
  {
    if( pl > 8 ) printf("x is NULL so testing if y is NULL...\n");
    return (Bool) ( y != (FMon) NULL );
  }
  else if( y == (FMon) NULL ) // If y is empty x cannot be less than it so just return 0
  {
    if( pl > 8 ) printf("y is NULL so returning 0...\n");
    return (Bool) 0;
  }
  else if ( fMonEqual( x, y ) == (Bool) 1 ) // If x == y just return 0
  {
    if( pl > 8 ) printf("x = y so returning 0...\n");
    return (Bool) 0;
  }
  else // Both non-empty and not equal
  {
    // Construct the generator for this iteration
    bigMon = ASCIIMon( (ULong) nOfGenerators - (ULong) bigVar + 1 );

    // Process x letter by letter, creating lists of intermediate terms
    while( fMonIsOne( x ) != (Bool) 1 )
    {
      xLetter = fMonPrefix( x, 1 ); // Look at the first letter
      if( fMonEqual( xLetter, bigMon ) == (Bool) 1 ) // if xLetter == bigMon
      {
        xCount++; // Increase the number of elements in the list
        xList = fMonListPush( xPad, xList );
        xPad = fMonOne(); // Reset
      }
      else
      {
```

```
      xPad = fMonTimes( xPad, xLetter ); // Build up next element
    }
    x = fMonSuffix( x, fMonLength( x ) − 1 ); // Look at next letter
  }
  xList = fMonListPush( xPad, xList ); // Flush out the remainder

  // Process y letter by letter
  while( fMonIsOne( y ) != (Bool) 1 )
  {
    yLetter = fMonPrefix( y, 1 ); // Look at the first letter
    if( fMonEqual( yLetter, bigMon ) == (Bool) 1 ) // if yLetter == bigMon
    {
      yCount++; // Increase the number of elements in the list
      yList = fMonListPush( yPad, yList );
      yPad = fMonOne(); // Reset
    }
    else
    {
      yPad = fMonTimes( yPad, yLetter ); // Build up next element
    }
    y = fMonSuffix( y, fMonLength( y ) − 1 ); // Look at next letter
  }
  yList = fMonListPush( yPad, yList ); // Flush out the remainder

  /*
   * Assuming representations
   * x = x0 z x1 z x2 ... z xn and
   * y = y0 z y1 z y2 ... z ym,
   *
   * We now have
   * xList = (xn, ..., x2, x1, x0),
   * yList = (ym, ..., y2, y1, y0),
   * and xCount and yCount hold the number of
   * z's in x and y respectively.
   *
   */

  // If xCount != yCount then we have a result...
  if( xCount < yCount )
  {
    if( pl > 8 ) printf("x has less of the highest weighted letter so returning 1...\n");
    return (Bool) 1;
  }
  else if( xCount > yCount )
  {
    if( pl > 8 ) printf("x has more of the highest weighted letter so returning 0...\n");
    return (Bool) 0;
  }
  else // ...otherwise we have to look at the intermediate terms
  {
    // Reverse the lists to obtain
    // xList = (x0, x1, x2, ..., xn) and
    // yList = (y0, y1, y2, ..., yn)
```

```
        xList = fMonListFXRev( xList );
        yList = fMonListFXRev( yList );

        // Increase the iteration value -- we will now compare the
        // elements of the lists w.r.t. the next highest variable
        bigVar++;
        while( xList )
        {
          i++;
          if( fMonWreathProd( xList -> first, yList -> first ) == (Bool) 1 )
          {
            if( pl > 8 ) printf("On component %u, x < y...\n", i);
            bigVar--; // reset before return
            return (Bool) 1;
          }
          else if( fMonWreathProd( yList -> first, xList -> first ) == (Bool) 1 )
          {
            if( pl > 8 ) printf("On component %u, y < x...\n", i);
            bigVar--; // reset before return
            return (Bool) 0;
          }
          else // (equal)
          {
            // Look at the next values in the sequence
            xList = xList -> rest;
            yList = yList -> rest;
          }
        }
        /*
         * Note: we should never reach this part of the code
         * because we know that at least one list comparison
         * will return a result (not all list comparisons will
         * return 'equal' because we know by this stage that
         * x is not equal to y). However we carry on for
         * completion.
         */
        bigVar--; // Reset
      }
    }

    printf("Executing Unreachable Code\n");
    exit( EXIT_FAILURE );
    return (Bool) 0;
}

/*
 * ==============================
 * Alphabet Manipulation Functions
 * ==============================
 */

/*
 * Function Name: preProcess
```

```
 *
 * Overview: Substitutes ASCII generators for original generators in a list of polynomials
 *
 * Detail: This function takes a list of polynomials _originalPolys_
 * in a set of generators _originalGenerators_ and returns the
 * same set of polynomials in ASCII generators, where the first
 * element of _originalGenerators_ is replaced by 'AAA', the
 * second element by 'AAB', etc.
 *
 * For example, if _originalGenerators_ = (x, y, z) so that the
 * generator order is x < y < z, and if _originalPolys_ = (x*y-z, 4*x^2-5*z),
 * the output list is (AAB*AAB-AAC, 4*AAA^2-5*AAC).
 *
 */
FAlgList
preProcess( originalPolys, originalGenerators )
FAlgList originalPolys;
FMonList originalGenerators;
{
  FAlgList newPolys = fAlgListNul;
  FAlg oldPoly, newPoly, adder;
  ULong i, oldPolySize, genLength, position;
  FMon firstTermMon, newFirstTermMon, multiplier, gen;
  QInteger firstTermCoef;

  // Go through each polynomial in turn...
  while( originalPolys )
  {
    oldPoly = originalPolys -> first; // Extract a polynomial
    originalPolys = originalPolys -> rest;
    oldPolySize = (ULong) fAlgNumTerms( oldPoly ); // Obtain the number of terms
    newPoly = fAlgZero(); // Initialise the new polynomial

    for( i = 1; i <= oldPolySize; i++ ) // For each term in the polynomial
    {
      firstTermMon = fAlgLeadMonom( oldPoly ); // Extract monomial
      firstTermCoef = fAlgLeadCoef( oldPoly ); // Extract coefficient
      oldPoly = fAlgReductum( oldPoly ); // Get ready to look at the next term
      newFirstTermMon = fMonOne(); // Initialise the new monomial

      // Go through each term replacing generators as required
      while( fMonIsOne( firstTermMon ) != (Bool) 1 )
      {
        gen = fMonPrefix( firstTermMon, 1 ); // Take the first letter 'x'
        position = fMonListPosition( gen, originalGenerators ); // Find the position of the letter in the list
        multiplier = ASCIIMon( position ); // Obtain the ASCII generator corresponding to x
        genLength = fMonLeadExp( firstTermMon ); // Find the exponent 'a' as in x^a
        // Multiply new monomial by (ASCII) x^a
        newFirstTermMon = fMonTimes( newFirstTermMon, fMonPow( multiplier, genLength ) );
        // Lose x^a from original monomial
        firstTermMon = fMonSuffix( firstTermMon, fMonLength( firstTermMon ) - genLength );
      }

```

```
      adder = fAlgMonom( firstTermCoef, newFirstTermMon ); // Construct the new ASCII term
      newPoly = fAlgPlus( newPoly, adder ); // Add the new ASCII term to the output polynomial
    }
    newPolys = fAlgListPush( newPoly, newPolys ); // Push new polynomial onto output list
  }

  // Return the reversed list (it was read in reverse)
  return fAlgListFXRev( newPolys );
}

/*
 * Function Name: postProcess
 *
 * Overview: Substitutes original generators for ASCII generators in a given polynomial
 *
 * Detail: This function takes a polynomial _oldPoly_ in ASCII generators
 * and returns the same polynomial in a corresponding set of generators
 * _originalGenerators_. The output is returned as a String
 * in fAlgToStr( ... ) format.
 *
 * For example, if _originalGenerators_ = (x, y, z) so that the
 * generator order is x < y < z, and if _oldPoly_ = A*B-C^2, then
 * the output String is "x y - z^2".
 *
 */
String
postProcess( oldPoly, originalGenerators )
FAlg oldPoly;
FMonList originalGenerators;
{
  FAlg adder;
  Bool result;
  FMon firstTermMon, gen, newFirstTermMon, multiplier;
  QInteger firstTermCoef;
  ULong i, match, oldPolySize, genLength;
  String back = strNew();

  sprintf( back, "" ); // Initialise back

  // Obtain the number of terms in the polynomial
  oldPolySize = (ULong) fAlgNumTerms( oldPoly );

  for( i = 1; i <= oldPolySize; i++ ) // For each term
  {
    firstTermMon = fAlgLeadMonom( oldPoly ); // Obtain the lead monomial
    firstTermCoef = fAlgLeadCoef( oldPoly ); // Obtain the lead coefficient
    result = qLess( firstTermCoef, qZero() ); // Test if coefficient is -ve
    oldPoly = fAlgReductum( oldPoly ); // Get ready to look at the next term
    newFirstTermMon = fMonOne(); // Initialise the new monomial

    // Go through the term replacing generators as required
    while( fMonIsOne( firstTermMon ) != (Bool) 1 )
    {
```

```
      gen = fMonPrefix( firstTermMon, 1 ); // Obtain the first letter 'x'
      genLength = fMonLeadExp( firstTermMon ); // Obtain 'a' as in x^a
      // Calculate the ASCII value ('AAA' = 1, 'AAB' = 2, ...)
      match = ASCIIVal( fMonToStr( gen ) );
      multiplier = fMonListNumber( match, originalGenerators ); // Find the original generator
      multiplier = fMonPow( multiplier, genLength );
      newFirstTermMon = fMonTimes( newFirstTermMon, multiplier ); // Multiply new monomial by the original x^a
      // Remove ASCII x^a from original monomial
      firstTermMon = fMonSuffix( firstTermMon, fMonLength( firstTermMon ) − genLength );
    }

    // Now add the term to the output string
    if( i == 1 ) // First term
      back = strConcat( back, fAlgToStr( fAlgMonom( firstTermCoef, newFirstTermMon ) ) );
    else // Must insert the correct sign (plus or minus)
    {
      if( result == 0 ) // Coefficient is +ve
      {
        adder = fAlgMonom( firstTermCoef, newFirstTermMon ); // Construct the new term
        back = strConcat( back, "␣+␣" );
        back = strConcat( back, fAlgToStr( adder ) );
      }
      else // Coefficient is −ve
      {
        adder = fAlgMonom( qNegate( firstTermCoef ), newFirstTermMon ); // Construct the new term
        back = strConcat( back, "␣-␣" );
        back = strConcat( back, fAlgToStr( adder ) );
      }
    }
  }

  return back;
}

/*
 * Function Name: postProcessParse
 *
 * Overview: As above but gives back its output in parse format
 *
 * Detail: This function takes a polynomial _oldPoly_ in ASCII generators
 * and returns the same polynomial in a corresponding set of generators
 * _originalGenerators_. The output is returned as a String in
 * parse format (with asterisks).
 *
 * For example, if _originalGenerators_ = (x, y, z) so that the
 * generator order is x < y < z, and if _oldPoly_ = A*B−C^2, then
 * the output String is "x*y − z^2".
 *
 */
String
postProcessParse( oldPoly, originalGenerators )
FAlg oldPoly;
FMonList originalGenerators;
```

```
{
  Short first = 1, written;
  FMon firstTermMon, gen, multiplier;
  QInteger firstTermCoef;
  ULong i, match, oldPolySize, genLength;
  String back = strNew();

  sprintf( back, "" ); // Initialise back

  if( !oldPoly ) // If input is NULL output the zero polynomial
  {
    back = strConcat( back, "0" );
    return back;
  }

  // Obtain the number of terms in the polynomial
  oldPolySize = (ULong) fAlgNumTerms( oldPoly );

  for( i = 1; i <= oldPolySize; i++ ) // For each term
  {
    // Assume to begin with that nothing has been added to
    // the String regarding the term we are now looking at
    written = 0;

    // Break down a term of the polynomial into its pieces
    firstTermMon = fAlgLeadMonom( oldPoly ); // Obtain the lead monomial
    firstTermCoef = fAlgLeadCoef( oldPoly ); // Obtain the lead coefficient

    if( qLess( firstTermCoef, qZero() ) == (Bool) 1 ) // If the coefficient is -ve
    {
      if( first == 1 ) // If this is the first term encountered
      {
        first = 0; // Set to avoid this loop in future

        // Note: there is no need for a space before the minus sign
        back = strConcat( back, "-" );
      }
      else // This is not the first term
      {
        // Separate two terms with a minus sign
        back = strConcat( back, " - " );
      }

      // Now that we have written the negative sign we can make
      // the coefficient positive
      firstTermCoef = qNegate( firstTermCoef );
    }
    else // The coefficient is +ve
    {
      if( first == 1 ) // If this is the first term encountered
      {
        first = 0; // Set to avoid this loop in future

```

```
      // Recall that there is no need to write out a plus
      // sign for the first term in a polynomial
    }
    else // This is not the first term
    {
      // Separate two terms with a plus sign
      back = strConcat( back, " + " );
    }
  }

  if( qIsOne( firstTermCoef ) != (Bool) 1 ) // If the coefficient is not one
  {
    written = 1; // Denote that we are going to write the coefficient to the String
    if( fMonEqual( firstTermMon, fMonOne() ) != (Bool) 1 ) // If the lead monomial is not 1
    {
      // Provide an asterisk to denote that the coefficient is
      // multiplied by the monomial
      back = strConcat( back, qToStr( firstTermCoef ) );
      back = strConcat( back, "*" );
    }
    else
    {
      // As the monomial is 1 there is no need to write the
      // monomial out and we can just write out the coefficient
      back = strConcat( back, qToStr( firstTermCoef ) );
    }
  }

  // If the lead monomial is not one
  if( fMonIsOne( firstTermMon ) != (Bool) 1 )
  {
    written = 1; // Denote that we are going to write the monomial to the String

    // Go through the term replacing generators as required
    while( firstTermMon )
    {
      gen = fMonPrefix( firstTermMon, 1 ); // Obtain the first letter 'x'
      genLength = fMonLeadExp( firstTermMon ); // Obtain 'a' as in x^a

      // Calculate the ASCII value ('AAA' = 1, 'AAB' = 2, ...)
      match = ASCIIVal( fMonToStr( gen ) );
      multiplier = fMonListNumber( match, originalGenerators ); // Find the original generator
      multiplier = fMonPow( multiplier, genLength );

      // Add multiplier onto the String
      back = strConcat( back, fMonToStr( multiplier ) );

      // Move the monomial onwards
      firstTermMon = fMonSuffix( firstTermMon, fMonLength( firstTermMon ) - genLength ); // Remove ASCII x^a
      if( firstTermMon ) back = strConcat( back, "*" );
    }
  }

```

```
    // If the coefficient is 1 and the monomial is 1 and nothing
    // has yet been written about this term, write "1" to the String
    // (This is to catch the case where the term is -1)
    if( ( qIsOne( firstTermCoef ) == (Bool) 1 )
        && ( fMonIsOne( firstTermMon ) == (Bool) 1 )
        && ( written == 0 ) )
    {
      back = strConcat( back, "1" );
    }

    oldPoly = fAlgReductum( oldPoly ); // Get ready to look at the next term
  }

  return back;
}

/*
 * Function Name: alphabetOptimise
 *
 * Overview: Adjusts the original generator order (1st arg) according to
 * frequency of generators in 2nd arg
 *
 * Detail: Given an FMonList _oldGens_ storing the given generator
 * order, this function optimises this order according to the
 * frequency of the generators in the polynomial list _polys_.
 * More specifically, the most frequently occurring generator
 * is set to be the smallest generator, the second most frequently
 * occurring generator is set to be the second smallest generator, ...
 * For the reasoning behind this optimisation, see a paper called
 * "A case where choosing a product order makes the
 * calculations of a Groebner basis much faster" by
 * Freyja Hreinsdottir (Journal of Symbolic Computation).
 *
 * Note: This function is designed to be used before the
 * generators and polynomials are converted to ASCII order.
 *
 * External variables needed: int pl;
 *
 */
FMonList
alphabetOptimise( oldGens, polys )
FMonList oldGens;
FAlgList polys;
{
  ULong i, j, letterLength, size = fMonListLength( oldGens ), scores[size];
  FMon monomial, letter, theLetters[size];
  FAlg poly;
  FMonList newGens = fMonListNul;

  if( pl > 0 )
  {
    printf("Old Ordering = ");
    fMonListDisplayOrder( oldGens );
```

```
  printf("\n");
}

// Set up arrays
for( i = 0; i < size; i++ )
{
  theLetters[i] = oldGens -> first; // Transfer generator to array
  oldGens = oldGens -> rest;
  scores[i] = 0; // Initialise scores
}

// Analyse the generators found in each polynomial
while( polys )
{
  poly = polys -> first; // Extract a polynomial
  if( pl > 2 ) printf("Counting generators in poly %s\n", fAlgToStr( poly ) );
  polys = polys -> rest;

  while( poly ) // For each term in the polynomial
  {
    monomial = fAlgLeadMonom( poly ); // Extract the lead monomial
    poly = fAlgReductum( poly );

    while( fMonIsOne( monomial ) != (Bool) 1 )
    {
      letter = fMonPrefix( monomial, 1 ); // Take the first letter 'x'
      letterLength = fMonLeadExp( monomial ); // Find the exponent 'a' as in x^a
      j = 0;
      while( j < size ) // Locate the letter in the generator array
      {
        if( fMonEqual( letter, theLetters[j] ) == (Bool) 1 )
        {
          // Match found, increase scores appropriately
          scores[j] = scores[j] + letterLength;
          j = size; // Shortcut search
        }
        else j++;
      }
      monomial = fMonSuffix( monomial, fMonLength( monomial ) - letterLength ); // Lose x^a from old monomial
    }
  }
}

if( pl > 0 ) // Provide some information on screen
{
  printf("Frequencies = ");
  for( i = 0; i < size; i++ )
  {
    printf("%u, ", scores[size-1-i] );
  }
  printf("\n");
}
```

```
  // Sort scores by a quicksort algorithm, adjusting the generators as we go along
  alphabetArrayQuickSort( scores, theLetters, 0, size−1 );

  // Build up new alphabet
  for( i = 1; i <= size; i++ )
    newGens = fMonListPush( theLetters[size−i], newGens );

  if( pl > 0 )
  {
    printf("New␣Ordering␣=␣");
    fMonListDisplayOrder( newGens );
    printf("\n");
  }

  // Return the sorted alphabet list
  return newGens;
}

/*
 * ==================================
 * Polynomial Manipulation Functions
 * ==================================
 */

/*
 * Function Name: fMonDiv
 *
 * Overview: Returns all possible ways that 2nd arg divides 1st arg;
 * 3rd arg = is division possible?
 *
 * Detail: Given two FMons _a_ and _b_, this function returns all possible
 * ways that _b_ divides _a_ in the form of an FMonPairList. The third
 * parameter _flag_ records whether or not (true/false) any divisions
 * are possible. For example, if t = abdababc and b = ab, then the
 * output FMonPairList is ((abdab, c), (abd, abc), (1, dababc)) and we
 * set _flag_ = true.
 *
 * External variables needed: int pl;
 *
 */
FMonPairList
fMonDiv( t, b, flag )
FMon t, b;
Short *flag;
{
  ULong i, tl, bl, diff;
  FMonPairList back = ( FMonPairList )theAllocFun( sizeof( *back ) );

  back = fMonPairListNul; // Initialise the output list

  *flag = false; // Assume there are no possible divisions to begin with
  tl = fMonLength( t );
  bl = fMonLength( b );
```

```
  if( tl < bl ) // There can be no possible divisions if |t| < |b|
  {
    return back;
  }
  else // Me must now consider each possibility in turn
  {
    diff = tl-bl;
    for( i = 0; i <= diff; i++ ) // Working left to right
    {
      // Is the subword of t of length |b| starting at position i+1 equal to b?
      if( fMonEqual( b, fMonSubWordLen( t, i+1, bl ) ) == (Bool) 1 )
      {
        // Match found; push the left and right factors onto the output list
        back = fMonPairListPush( fMonPrefix( t, i ), fMonSuffix( t, tl-bl-i ), back );
        if( pl > 6 ) printf("i = %i: %s = %s*(%s)*%s\n", i+1, fMonToStr( t ),
                        fMonToStr( fMonPrefix( t, i ) ), fMonToStr( b ), fMonToStr( fMonSuffix( t, tl-bl-i ) ) );
      }
    }
  }

  // If we found some matches set _flag_ to be true
  if( back ) *flag = true;
  return back; // Return the output list
}

/*
 * Function Name: fMonDivFirst
 *
 * Overview: Returns the first way that 2nd arg divides 1st arg;
 * 3rd arg = is division possible?
 *
 * Detail: Given two FMons _a_ and _b_, this function returns the first
 * way that _b_ divides _a_ in the form of an FMonPairList. The third
 * parameter _flag_ records whether or not (true/false) any divisions
 * are possible. For example, if t = abdababc and b = ab, then the
 * output FMonPairList is ((1, dababc)) and we
 * set _flag_ = true.
 *
 * External variables needed: int pl;
 *
 */
FMonPairList
fMonDivFirst( t, b, flag )
FMon t, b;
Short *flag;
{
  ULong i, tl, bl, diff;
  FMonPairList back = ( FMonPairList )theAllocFun( sizeof( *back ));

  back = fMonPairListNul; // Initialise the output list

  *flag = false; // Assume there are no possible divisions to begin with
```

```
  tl = fMonLength( t );
  bl = fMonLength( b );

  if( tl < bl ) // There can be no possible divisions if |t| < |b|
  {
    return back;
  }
  else // Me must now consider each possibility in turn
  {
    diff = tl-bl;
    for( i = 0; i <= diff; i++ ) // Working left to right
    {
      // Is the subword of t of length |b| starting at position i+1 equal to b?
      if( fMonEqual( b, fMonSubWordLen( t, i+1, bl ) ) == (Bool) 1 )
      {
        // Match found; push the left and right factors onto the output list and return it
        back = fMonPairListPush( fMonPrefix( t, i ), fMonSuffix( t, tl-bl-i ), back );
        if( pl > 6 ) printf("i = %i: %s = %s*(%s)*%s\n", i+1, fMonToStr( t ),
                     fMonToStr( fMonPrefix( t, i ) ), fMonToStr( b ), fMonToStr( fMonSuffix( t, tl-bl-i ) ) );
        *flag = true; // Indicate that we have found a match
        return back;
      }
    }
  }

  return back; // Return the empty output list - no matches were found
}

/*
 * Function Name: fMonOverlaps
 *
 * Overview: Finds all possible overlaps of 2 FMons
 *
 * Detail: Given two FMons, this function returns all
 * possible ways in which the two monomials overlap.
 * For example, if _a_ = abcabc and _b_ = cab, then
 * the output FMonPairList is
 * ((1, 1), (ab, c), (c, 1), (1, cabc), (1, ab), (abcab, 1))
 * as in
 * 1*(abcabc)*1 = ab*(cab)*c,
 * c*(abcabc)*1 = 1*(cab)*cabc,
 * 1*(abcabc)*ab = abcab*(cab)*1.
 *
 * External variables needed: int pl;
 *
 */
FMonPairList
fMonOverlaps( a, b )
FMon a, b;
{
  FMon still, move;
  Short type;
  ULong la, lb, ls, lm, i;
```

```
FMonPairList back = ( FMonPairList )theAllocFun( sizeof( *back ));

back = fMonPairListNul; // Initialise the output list

la = fMonLength( a );
lb = fMonLength( b );

// Check for the trivial monomial
if( ( la == 0 ) || ( lb == 0 ) ) return back;

// Determine which monomial has the greater length
if( la < lb )
{
  still = b; ls = lb;
  move = a; lm = la;
  type = 1; // Remember that |a| < |b|
}
else
{
  still = a; ls = la;
  move = b; lm = lb;
  type = 2; // Remember that |a| >= |b|
}

// First deal with prefix and suffix overlaps
for( i = 1; i <= lm-1; i++ )
{
  // PREFIX overlap - is a prefix of still equal to a suffix of move?
  if( fMonEqual( fMonPrefix( still, i ), fMonSuffix( move, i ) ) == (Bool) 1 )
  {
    if( type == 1 ) // still = b, move = a
    {
      // Need to multiply a on the right and b on the left to construct the overlap
      back = fMonPairListPush( fMonPrefix( a, la-i ), fMonOne(), back ); // b
      back = fMonPairListPush( fMonOne(), fMonSuffix( b, lb-i ), back ); // a
      if( pl > 5 ) printf("Left Overlap Found for (%s, %s): (%s, %s, %s, %s)\n",
                          fMonToStr( a ), fMonToStr( b ), fMonToStr( fMonOne() ),
                          fMonToStr( fMonSuffix( b, lb-i ) ), fMonToStr( fMonPrefix( a, la-i ) ),
                          fMonToStr( fMonOne() ) );
    }
    else // still = a, move = b
    {
      // Need to multiply a on the left and b on the right to construct the overlap
      back = fMonPairListPush( fMonOne(), fMonSuffix( a, la-i ), back ); // b
      back = fMonPairListPush( fMonPrefix( b, lb-i ), fMonOne(), back ); // a
      if( pl > 5 ) printf("Left Overlap Found for (%s, %s): (%s, %s, %s, %s)\n",
                          fMonToStr( a ), fMonToStr( b ), fMonToStr( fMonPrefix( b, lb-i ) ),
                          fMonToStr( fMonOne() ), fMonToStr( fMonOne() ),
                          fMonToStr( fMonSuffix( a, la-i ) ) );
    }
  }

  // SUFFIX overlap - is a suffix of still equal to a prefix of move?
```

```
  if( fMonEqual( fMonSuffix( still, i ), fMonPrefix( move, i ) ) == (Bool) 1 )
  {
    if( type == 1 ) // still = b, move = a
    {
      // Need to multiply a on the left and b on the right to construct the overlap
      back = fMonPairListPush( fMonOne(), fMonSuffix( a, la-i ), back ); // b
      back = fMonPairListPush( fMonPrefix( b, lb-i ), fMonOne(), back ); // a
      if( pl > 5 ) printf("Right Overlap Found for (%s, %s): (%s, %s, %s, %s)\n",
                      fMonToStr( a ), fMonToStr( b ), fMonToStr( fMonPrefix( b, lb-i ) ),
                      fMonToStr( fMonOne() ), fMonToStr( fMonOne() ),
                      fMonToStr( fMonSuffix( a, la-i ) ) );
    }
    else // still = a, move = b
    {
      // Need to multiply a on the right and b on the left to construct the overlap
      back = fMonPairListPush( fMonPrefix( a, la-i ), fMonOne(), back ); // b
      back = fMonPairListPush( fMonOne(), fMonSuffix( b, lb-i ), back ); // a
      if( pl > 5 ) printf("Right Overlap Found for (%s, %s): (%s, %s, %s, %s)\n",
                      fMonToStr( a ), fMonToStr( b ), fMonToStr( fMonOne() ),
                      fMonToStr( fMonSuffix( b, lb-i ) ), fMonToStr( fMonPrefix( a, la-i ) ),
                      fMonToStr( fMonOne() ) );
    }
  }
}

// Subword overlaps
for( i = 1; i <= ls-lm+1; i++ )
{
  if( fMonEqual( move, fMonSubWordLen( still, i, lm ) ) == (Bool) 1 )
  {
    if( type == 1 ) // still = b, move = a
    {
      // Need to multiply a on the left and right to construct the overlap
      back = fMonPairListPush( fMonOne(), fMonOne(), back ); // b
      back = fMonPairListPush( fMonPrefix( b, i-1 ), fMonSuffix( b, lb+1-i-lm ), back ); // a
      if( pl > 5 ) printf("Middle Overlap Found for (%s, %s): (%s, %s, %s, %s)\n",
                      fMonToStr( a ), fMonToStr( b ), fMonToStr( fMonPrefix( b, i-1 ) ),
                      fMonToStr( fMonSuffix( b, lb+1-i-lm ) ), fMonToStr( fMonOne() ),
                      fMonToStr( fMonOne() ) );
    }
    else // still = a, move = b
    {
      // Need to multiply b on the left and right to construct the overlap
      back = fMonPairListPush( fMonPrefix( a, i-1 ), fMonSuffix( a, la+1-i-lm ), back ); // b
      back = fMonPairListPush( fMonOne(), fMonOne(), back ); // a
      if( pl > 5 ) printf("Middle Overlap Found for (%s, %s): (%s, %s, %s, %s)\n",
                      fMonToStr( a ), fMonToStr( b ), fMonToStr( fMonOne() ),
                      fMonToStr( fMonOne() ), fMonToStr( fMonPrefix( a, i-1 ) ),
                      fMonToStr( fMonSuffix( a, la+1-i-lm ) ) );
    }
  }
}
```

```
  return back;
}

/*
 * Function Name: degInitial
 *
 * Overview: Returns the degree-based initial of a given polynomial
 *
 * Detail: Given a polynomial _input_, this function returns the
 * initial of that polynomial w.r.t. degree. In other words,
 * all terms of highest degree are returned.
 *
 */
FAlg
degInitial( input )
FAlg input;
{
  FAlg output = fAlgZero();
  ULong max = 0, next;

  // If the input is trivial, the output is trivial
  if( !input ) return input;

  // For each term in the input polynomial
  while( input )
  {
    // Find the degree of the next term in the polynomial
    next = fMonLength( fAlgLeadMonom( input ) );

    // If we find a term of higher degree
    if( next > max )
    {
      // Set a new maximum
      max = next;
      // Start building up the output polynomial again
      output = fAlgLeadTerm( input );
    }
    // Else if we find a term of equal maximum degree
    else if( next == max )
    {
      // Add the term to the output polynomial
      output = fAlgPlus( output, fAlgLeadTerm( input ) );
    }

    // Get ready to look at the next term of the input polynomial
    input = fAlgReductum( input );
  }

  // Return the initial
  return output;
}

/*
```

```
 * Function Name: fMonReverse
 *
 * Overview: Reverses a monomial
 *
 * Detail: Given a monomial m = x_1x_2...x_n, this
 * function returns the monomial m' = x_nx_{n-1}...x_2x_1.
 *
 */
FMon
fMonReverse( input )
FMon input;
{
  FMon output = fMonOne();

  // For each variable in the input monomial
  while( input )
  {
    output = fMonTimes( fMonPrefix( input, 1 ), output );
    input = fMonRest( input );
  }

  // Return the reversed monomial
  return output;
}

/*
 * ========================
 * Groebner Basis Functions
 * ========================
 */

/*
 * Function Name: polyReduce
 *
 * Overview: Returns the normal form of a polynomial w.r.t. a list of polynomials
 *
 * Detail: Given an FAlg and an FAlgList, this function
 * divides the FAlg w.r.t. the FAlgList, returning the
 * normal form of the input polynomial w.r.t. the list.
 *
 * External Variables Required: int pl;
 * Global Variables Used: ULong nRed;
 *
 */
FAlg
polyReduce( poly, list )
FAlg poly;
FAlgList list;
{
  ULong i, numRules = fAlgListLength( list );
  FAlg back = fAlgZero(), lead, upgrade, LHSA[numRules];
  FMon leadMonomial, leadLoopMonomial, LHSM[numRules];
  FMonPairList factors;
```

```
QInteger leadQ, leadLoopQ, lcmQ, LHSQ[numRules];
Short flag, toggle;

// Convert the input list of polynomials to an array and
// create arrays of lead monomials and lead coefficients
for( i = 0; i < numRules; i++ )
{
  if( pl > 5 ) printf("Poly %u = %s\n", i+1, fAlgToStr( list -> first ) );
  LHSA[i] = list -> first;
  LHSM[i] = fAlgLeadMonom( list -> first );
  LHSQ[i] = fAlgLeadCoef( list -> first );
  list = list -> rest;
}

// We will now recursively reduce every term in the polynomial
// until no more reductions are possible
while( fAlgIsZero( poly ) != (Bool) 1 )
{
  if( pl > 5 ) printf("Looking at Lead Term of %s\n", fAlgToStr( poly ) );
  toggle = 1; // Assume no reductions are possible to begin with
  lead = fAlgLeadTerm( poly );
  leadMonomial = fAlgLeadMonom( lead );
  leadQ = fAlgLeadCoef( lead );
  i = 0;

  while( i < numRules ) // For each polynomial in the list
  {
    leadLoopMonomial = LHSM[i]; // Pick a test monomial
    flag = false;
    // Does the ith polynomial divide our polynomial?
    factors = fMonDivFirst( leadMonomial, leadLoopMonomial, &flag );

    if( flag == true ) // i.e. leadMonomial = factors -> lft * leadLoopMonomial * factors -> rt
    {
      if( pl > 1 ) nRed++; // Increase the number of reductions carried out
      if( pl > 5 ) printf("Found %s = (%s) * (%s) * (%s)\n", fMonToStr( leadMonomial ),
                          fMonToStr( factors -> lft ), fMonToStr( leadLoopMonomial ),
                          fMonToStr( factors -> rt ) );
      toggle = 0; // Indicate a reduction has been carried out to exit the loop
      leadLoopQ = LHSQ[i]; // Pick the divisor's leading coefficient
      lcmQ = AltLCMQInteger( leadQ, leadLoopQ ); // Pick 'nice' cancelling coefficients

      // Construct poly #i * -1 * coefficient to get lead terms the same
      upgrade = fAlgTimes( fAlgMonom( qOne(), factors -> lft ), LHSA[i] );
      upgrade = fAlgTimes( upgrade, fAlgMonom( qNegate( qDivide( lcmQ, leadLoopQ ) ), factors -> rt ) );

      // Add in poly * coefficient to cancel off the lead terms
      upgrade = fAlgPlus( upgrade, fAlgScaTimes( qDivide( lcmQ, leadQ ), poly ) );

      // We must also now multiply the current discarded remainder by a factor
      back = fAlgScaTimes( qDivide( lcmQ, leadQ ), back );
      poly = upgrade; // In the next iteration, we will be reducing the new polynomial upgrade
      if( pl > 5 ) printf("New Word = %s; New Remainder = %s\n", fAlgToStr( poly ), fAlgToStr( back ) );
```

```
      }
      if( toggle == 1 ) // The ith polynomial did not divide poly
        i++;
      else // A reduction was carried out, exit the loop
        i = numRules;
    }

    if( toggle == 1 ) // No reductions were carried out; now look at the next term
    {
      // Add lead term to remainder and reduce the rest of the polynomial
      back = fAlgPlus( back, lead );
      poly = fAlgReductum( poly );
      if( pl > 5 ) printf("New Remainder = %s\n", fAlgToStr( poly ) );
    }
  }

  return back; // Return the reduced and simplified polynomial
}

/*
 * Function Name: minimalGB
 *
 * Overview: Minimises a given Groebner Basis
 *
 * Detail: Given an input Groebner Basis, this function
 * will eliminate from the basis any polynomials whose
 * lead monomials are multiples of some other lead
 * monomial.
 *
 * External variables required: int pl;
 *
 */
FAlgList
minimalGB( G )
FAlgList G;
{
  FAlgList G_Minimal = fAlgListNul, G_Copy = fAlgListCopy( G );
  ULong i, p, length = fAlgListLength( G );
  FMon checker[length];
  FMonPairList sink;
  Short flag, blackList[length];

  // Create an array of lead monomials and initialise blackList
  // which will store which monomials are to be deleted from the basis
  for( i = 0; i < length; i++ )
  {
    blackList[i] = 0;
    checker[i] = fAlgLeadMonom( G_Copy -> first );
    G_Copy = G_Copy -> rest;
  }

  // Test divisibility of each monomial w.r.t all other monomials
  for( i = 0; i < length; i++ )
```

```
  {
    p = 0;
    while( p < length )
    {
      // If p is different from i and p has not yet been 'deleted' from the basis
      if( ( p != i ) && ( blackList[p] != 1 ) )
      {
        flag = false;
        sink = fMonDiv( checker[i], checker[p], &flag );
        if( flag == true ) // poly i's lead term is a multiple of poly p's lead term
        {
          blackList[i] = 1; // Ensure polynomial i is deleted later on
          break; // Exit from the while loop
        }
      }
      p++;
    }
  }

  // Push onto the output list elements not blacklisted
  for( i = 0; i < length; i++ )
  {
    if( blackList[i] == 0 ) // Not to be deleted
    {
      G_Minimal = fAlgListPush( G -> first, G_Minimal );
    }
    G = G -> rest; // Advance the list
  }

  // As it was constructed in reverse, we must reverse G_Minimal before returning it
  return fAlgListFXRev( G_Minimal );
}

/*
 * Function Name: reducedGB
 *
 * Overview: Reduces each member of a Groebner Basis w.r.t. all other members
 *
 * Detail: Given a list of polynomials, this function takes each
 * member of the list in turn, reducing the polynomial w.r.t. all
 * other members of the basis.
 *
 * Note: This function does not check whether a polynomial reduces to
 * zero or not (we usually want to delete polynomials that reduce to
 * zero from our basis) - it is assumed that no member of the basis will
 * reduce to zero (which will be the case if we start with a minimal Groebner
 * Basis). Also, at the end of the algorithm, the total number of reductions
 * carried out during the *whole program* is reported if the print level
 * (pl) exceeds 1.
 *
 * External variables required: int pl;
 *
 */
```

```
FAlgList
reducedGB( GBasis )
FAlgList GBasis;
{
  FAlg poly;
  FAlgList back = fAlgListNul, old_G, new_G;
  ULong i, sizeOfInput = fAlgListLength( GBasis );

  if( sizeOfInput > 1 ) // If |GBasis| > 1
  {
    i = 0; // i keeps track of which polynomial we are looking at

    // Start by making a copy of G for processing
    old_G = fAlgListCopy( GBasis );

    while( old_G ) // For each polynomial
    {
      i++;
      poly = old_G -> first; // Extract a polynomial
      old_G = old_G -> rest; // Advance the list
      if( pl > 2 ) printf("\nLooking at element p = %s of basis\n", fAlgToStr( poly ) );

      // Construct basis without 'poly' by appending
      // the remaining polynomials to the reduced polynomials
      new_G = fAlgListAppend( back, old_G );

      poly = polyReduce( poly, new_G ); // Reduce poly w.r.t. new_G
      poly = findGCD( poly ); // Divide out by the GCD

      if( pl > 2 ) printf("Reduced p to %s\n", fAlgToStr( poly ) );
      // Add the reduced polynomial to the list
      back = fAlgListAppend( back, fAlgListSingle( poly ) );
    }
  }
  else // else |GBasis| = 1 and there is no point in doing any reduction
  {
    return GBasis;
  }

  // Report on the total number of reductions carried out during the *whole program*
  if( pl > 1 ) printf("Number of Reductions Carried out = %u\n", nRed );

  return back;
}

/*
 * Function Name: idealMembershipProblem
 *
 * Overview: Tests whether a given FAlg reduces to 0 using the given FAlgList
 *
 * Detail: Given a list of polynomials, this function tests whether
 * a given polynomial reduces to zero using this list. This is
 * done using a modified version of the function polyReduce in that
```

```
 * the moment an irreducible monomial is encountered, the algorithm
 * terminates with the knowledge that the polynomial will not
 * reduce to 0.
 *
 * External variables required: int pl;
 *
 */
Bool
idealMembershipProblem( poly, list )
FAlg poly;
FAlgList list;
{
  ULong i, numRules = fAlgListLength( list );
  FAlg back = fAlgZero(), lead, upgrade, LHSA[numRules];
  FMon leadMonomial, leadLoopMonomial, LHSM[numRules];
  FMonPairList factors;
  QInteger leadQ, leadLoopQ, lcmQ, LHSQ[numRules];
  Short flag, toggle;

  // Convert the input list of polynomials to an array and
  // create arrays of lead monomials and lead coefficients
  for( i = 0; i < numRules; i++ )
  {
    if( pl > 5 ) printf("Poly %u = %s\n", i+1, fAlgToStr( list -> first ) );
    LHSA[i] = list -> first;
    LHSM[i] = fAlgLeadMonom( list -> first );
    LHSQ[i] = fAlgLeadCoef( list -> first );
    list = list -> rest;
  }

  // We will now recursively reduce every term in the polynomial
  // until an irreducible term is encountered or no more reductions are possible
  while( fAlgIsZero( poly ) != (Bool) 1 )
  {
    if( pl > 5 ) printf("Looking at Lead Term of %s\n", fAlgToStr( poly ) );
    toggle = 1; // Assume no reductions are possible to begin with
    lead = fAlgLeadTerm( poly );
    leadMonomial = fAlgLeadMonom( lead );
    leadQ = fAlgLeadCoef( lead );
    i = 0;

    while( i < numRules ) // For each polynomial in the list
    {
       leadLoopMonomial = LHSM[i]; // Pick a test monomial
       flag = false;
       // Does the ith polynomial divide our polynomial?
       factors = fMonDivFirst( leadMonomial, leadLoopMonomial, &flag );

       if( flag == true ) // i.e. leadMonomial = factors -> lft * leadLoopMonomial * factors -> rt
       {
          if( pl > 5 ) printf("Found %s = (%s) * (%s) * (%s)\n", fMonToStr( leadMonomial ),
                                      fMonToStr( factors -> lft ), fMonToStr( leadLoopMonomial ),
                                      fMonToStr( factors -> rt ) );
```

```
            toggle = 0; // Indicate a reduction has been carried out to exit the loop
            leadLoopQ = LHSQ[i]; // Pick the divisor's leading coefficient
            lcmQ = AltLCMQInteger( leadQ, leadLoopQ ); // Pick 'nice' cancelling coefficients

            // Construct poly #i * -1 * coefficient to get lead terms the same
            upgrade = fAlgTimes( fAlgMonom( qOne(), factors -> lft ), LHSA[i] );
            upgrade = fAlgTimes( upgrade, fAlgMonom( qNegate( qDivide( lcmQ, leadLoopQ ) ), factors -> rt ) );

            // Add in poly * coefficient to cancel off the lead terms
            upgrade = fAlgPlus( upgrade, fAlgScaTimes( qDivide( lcmQ, leadQ ), poly ) );

            // We must also now multiply the current discarded remainder by a factor
            back = fAlgScaTimes( qDivide( lcmQ, leadQ ), back );
            poly = upgrade; // In the next iteration, we will be reducing the new polynomial upgrade
            if( pl > 5 ) printf("New Word = %s; New Remainder = %s\n", fAlgToStr( poly ), fAlgToStr( back ) );
        }
        if( toggle == 1 ) // The ith polynomial did not divide poly
          i++;
        else // A reduction was carried out, exit the loop
          i = numRules;
      }

      /*
       * If toggle == 1, this means that no rule simplified the lead term of 'poly'
       * so that we have encountered an irreducible monomial. In this case, the polynomial
       * we are reducing will not reduce to zero, so we can now return 0.
       */
      if( toggle == 1 )
        return (Bool) 0;
    }

  // If we reach here, the polynomial reduced to 0 so we return a positive result.
  return (Bool) 1;
}

/*
 * ===========
 * End of File
 * ===========
 */
```

B.2.8 list_functions.h

```
/*
 * File: list_functions.h
 * Author: Gareth Evans
 * Last Modified: 9th August 2005
 */

// Initialise file definition
# ifndef LIST_FUNCTIONS_HDR
# define LIST_FUNCTIONS_HDR
```

```

// Include MSSRC Libraries
# include <fralg.h>

//
// External Variables Required
//

extern int pl; // Holds the "Print Level"

//
// Display Functions
//

// Displays an FMonList in the format l1\n l2\n l3\n...
void fMonListDisplay( FMonList );
// Displays an FMonList in the format l1 > l2 > l3...
void fMonListDisplayOrder( FMonList );
// Displays an FMonPairList in the format (l1, l2)\n (l3, l4)\n...
void fMonPairListMultDisplay( FMonPairList );
// Displays an FAlgList in the format p1\n p2\n p3\n...
void fAlgListDisplay( FAlgList );

//
// List Extraction Functions
//

// Returns the ith member of an FMonList (i = 1st arg)
FMon fMonListNumber( ULong, FMonList );
// Returns the ith member of an FMonPairList (i = 1st arg)
FMonPair fMonPairListNumber( ULong, FMonPairList );
// Returns the ith member of an FAlgList (i = 1st arg)
FAlg fAlgListNumber( ULong, FAlgList );

//
// List Membership Functions
//

// Does the FAlg appear in the FAlgList? (1 = yes)
Bool fAlgListIsMember( FAlg, FAlgList );

//
// List Position Functions
//

// Gives position of 1st appearance of FMon in FMonList
ULong fMonListPosition( FMon, FMonList );
// Gives position of 1st appearance of FAlg in FAlgList
ULong fAlgListPosition( FAlg, FAlgList );

//
// Sorting Functions
//
```

```
// Swaps 2 elements in arrays of ULongs and FMons
void alphabetArraySwap( ULong[], FMon[], ULong, ULong );
// Sorts an array of ULongs (largest first) and applies the same changes to the FMon array
void alphabetArrayQuickSort( ULong[], FMon[], ULong, ULong );
// Swaps 2 elements in arrays of FAlgs and FMons
void fAlgArraySwap( FAlg[], FMon[], ULong, ULong );
// Sorts an array of FAlgs using DegRevLex (largest first)
void fAlgArrayQuickSortDRL( FAlg[], FMon[], ULong, ULong );

// Sorts an array of FAlgs using theOrdFun (largest first)
void fAlgArrayQuickSortOrd( FAlg[], FMon[], ULong, ULong );
// Sorts an FAlgList (largest first)
FAlgList fAlgListSort( FAlgList, int );
// Swaps 2 elements in arrays of FMons, ULongs and ULongs
void multiplicativeArraySwap( FMon[], ULong[], ULong[], ULong, ULong );
// Sorts input data to OverlapDiv w.r.t. DegRevLex (largest first)
void multiplicativeQuickSort( FMon[], ULong[], ULong[], ULong, ULong );

//
// Insertion Sort Functions
//

// Insert into list according to DegRevLex
FAlgList fAlgListDegRevLexPush( FAlg, FAlgList );
// As above, but also returns the insertion position
FAlgList fAlgListDegRevLexPushPosition( FAlg, FAlgList, ULong * );
// Insert into list according to the current monomial ordering
FAlgList fAlgListNormalPush( FAlg, FAlgList );
// As above, but also returns the insertion position
FAlgList fAlgListNormalPushPosition( FAlg, FAlgList, ULong * );

//
// Deletion Functions
//

// Removes the (1st arg)-th element from the list
FMonList fMonListRemoveNumber( ULong, FMonList );
// Removes the (1st arg)-th element from the list
FMonPairList fMonPairListRemoveNumber( ULong, FMonPairList );
// Removes the (1st arg)-th element from the list
FAlgList fAlgListRemoveNumber( ULong, FAlgList );

//
// Normalising Functions
//

// Removes any fractions found in the FAlgList by scalar multiplication
FAlgList fAlgListRemoveFractions( FAlgList );

# endif // LIST_FUNCTIONS_HDR
```

B.2.9 list_functions.c

```
/*
 * File: list_functions.c
 * Author: Gareth Evans, Chris Wensley
 * Last Modified: 9th August 2005
 */

/*
 * =================
 * Display Functions
 * =================
 */

/*
 * Function Name: fMonListDisplay
 *
 * Overview: Displays an FMonList in the format l1\n l2\n l3\n...
 *
 * Detail: Given an FMonList, this function displays the
 * elements of the list on screen in such a way that if the
 * list is (for example) L = (l1, l2, l3, l4), the output is
 *
 * l1
 * l2
 * l3
 * l4
 *
 */
void
fMonListDisplay( L )
FMonList L;
{
  while( L )
  {
    printf( "%s\n", fMonToStr( L -> first ) );
    L = L -> rest;
  }
}

/*
 * Function Name: fMonListDisplayOrder
 *
 * Overview: Displays an FMonList in the format l1 < l2 < l3...
 *
 * Detail: Given an FMonList, this function displays the
 * elements of the list on screen in such a way that if the
 * list is (for example) L = (l1, l2, l3, l4), the output is
 *
 * l4 > l3 > l2 > l1
 *
 * External variables required: int pl;
 *
```

```
 */
void
fMonListDisplayOrder( L )
FMonList L;
{
  ULong i = 1, j = fMonListLength( L );

  L = fMonListRev( L );

  while( L )
  {
    if( pl >= 1 )
      printf( "%s", fMonToStr( L -> first ) );
    if( pl > 1 )
    {
      printf( " (%s)", ASCIIStr( j + 1 - i ) );
      i++;
    }
    L = L -> rest;
    if( L ) // If there is another element left provide a " > "
    {
      printf( " > " );
    }
  }
}

/*
 * Function Name: fMonPairListMultDisplay
 *
 * Overview: Displays an FMonPairList in the format (l1, l2)\n (l3, l4)\n...
 *
 * Detail: Given an FMonPairList, this function displays the
 * elements of the list on screen in such a way that if the
 * list is (for example) L = ((l1, l2), (l3, l4), (l5, l6)), the output is
 *
 * (l1, l2)
 * (l3, l4)
 * (l5, l6)
 *
 * Remark: The "Mult" stands for multiplicative - this function is primarily
 * used to display (Left, Right) multiplicative variables.
 */
void
fMonPairListMultDisplay( L )
FMonPairList L;
{
  while( L )
  {
    printf( "(%s, %s)\n", fMonToStr( L -> lft ), fMonToStr( L -> rt ) );
    L = L -> rest;
  }
}
```

```
/*
 * Function Name: fAlgListDisplay
 *
 * Overview: Displays an FAlgList in the format p1\n p2\n p3\n...
 *
 * Detail: Given an FAlgList, this function displays the
 * elements of the list on screen in such a way that if the
 * list is (for example) L = (p1, p2, p3, p4), the output is
 *
 * p1
 * p2
 * p3
 * p4
 *
 */
void
fAlgListDisplay( L )
FAlgList L;
{
  while( L )
  {
    printf( "%s\n", fAlgToStr( L -> first ) );
    L = L -> rest;
  }
}

/*
 * ========================
 * List Extraction Functions
 * ========================
 */

/*
 * Function Name: fMonListNumber
 *
 * Overview: Returns the ith member of an FMonList (i = 1st arg)
 *
 * Detail: Given an FMonList, this function returns the
 * ith member of that list, where i is the first argument _number_.
 *
 */
FMon
fMonListNumber( number, list )
ULong number;
FMonList list;
{
  ULong i;
  FMon back = newFMon();

  for( i = 1; i < number; i++ )
  {
    list = list -> rest; // Traverse list
  }
```

```

  back = list -> first;
  return back;
}

/*
 * Function Name: fMonPairListNumber
 *
 * Overview: Returns the ith member of an FMonPairList (i = 1st arg)
 *
 * Detail: Given an FMonPairList, this function returns the
 * ith member of that list, where i is the first argument _number_.
 *
 */
FMonPair
fMonPairListNumber( number, list )
ULong number;
FMonPairList list;
{
  FMonPair back;
  ULong i;

  for( i = 1; i < number; i++ )
  {
    list = list -> rest; // Traverse list
  }

  back.lft = list -> lft;
  back.rt = list -> rt;

  return back;
}

/*
 * Function Name: fAlgListNumber
 *
 * Overview: Returns the ith member of an FAlgList (i = 1st arg)
 *
 * Detail: Given an FAlgList, this function returns the
 * ith member of that list, where i is the first argument _number_.
 *
 */
FAlg
fAlgListNumber( number, list )
ULong number;
FAlgList list;
{
  ULong i;
  FAlg back = newFAlg();

  for( i = 1; i < number; i++ )
  {
    list = list -> rest; // Traverse list
```

```
  }

  back = list -> first;
  return back;
}

/*
 * ========================
 * List Membership Functions
 * ========================
 */

/*
 * Function Name: fAlgListIsMember
 *
 * Overview: Does the FAlg appear in the FAlgList? (1 = yes)
 *
 * Detail: Given an FAlgList, this function tests whether
 * a given FAlg appears in the list. This is done by
 * moving through the list and checking each entry
 * sequentially. Once a match is found, a positive result
 * is returned; otherwise once we have gone through the
 * entire list, a negative result is returned.
 *
 */
Bool
fAlgListIsMember( w, L )
FAlg w;
FAlgList L;
{
  while( L )
  {
    if( fAlgEqual( w, L -> first ) == (Bool) 1 )
    {
      return (Bool) 1; // Match found
    }
    L = L -> rest;
  }
  return (Bool) 0; // No matches found
}

/*
 * ======================
 * List Position Functions
 * ======================
 */

/*
 * Function Name: fMonListPosition
 *
 * Overview: Gives position of 1st appearance of FMon in FMonList
 *
 * Detail: Given an FMonList, this function returns the
```

```
 * position of the first appearance of a given FMon in that
 * list. If the FMon does not appear in the list,
 * 0 is returned.
 *
 */
ULong
fMonListPosition( w, L )
FMon w;
FMonList L;
{
  ULong pos = 0; // Current position in list

  if( fMonListLength( L ) == 0 )
  {
    return (ULong) 0; // List is empty so no match
  }
  while( L ) // While there are still elements in the list
  {
    pos++;
    if( fMonEqual( w, L -> first ) == (Bool) 1 )
    {
      return pos; // Match found; return position
    }
    L = L -> rest;
  }
  return (ULong) 0; // No match found in the list
}

/*
 * Function Name: fAlgListPosition
 *
 * Overview: Gives position of 1st appearance of FAlg in FAlgList
 *
 * Detail: Given an FAlgList, this function returns the
 * position of the first appearance of a given FAlg in that
 * list. If the FAlg does not appear in the list, 0 is returned.
 *
 */
ULong
fAlgListPosition( w, L )
FAlg w;
FAlgList L;
{
  ULong pos = 0; // Current position in list

  if( fAlgListLength( L ) == 0 )
  {
    return (ULong) 0; // List is empty so no match
  }
  while( L ) // While there are still elements in the list
  {
    pos++;
    if( fAlgEqual( w, L -> first ) == (Bool) 1 )
```

```
    {
      return pos; // Match found; return position
    }
    L = L -> rest;
  }
  return (ULong) 0; // No match found in the list
}

/*
 * =================
 * Sorting Functions
 * =================
 */

/*
 * Function Name: alphabetArraySwap
 *
 * Overview: Swaps 2 elements in arrays of ULongs and FMons
 *
 * Detail: Given an array of ULongs and an array of FMons,
 * this function swaps the ith and jth elements of both arrays.
 *
 */
void
alphabetArraySwap( array1, array2, i, j )
ULong array1[];
FMon array2[];
ULong i, j;
{
  ULong swap1;
  FMon swap2 = newFMon();

  swap1 = array1[i];
  swap2 = array2[i];
  array1[i] = array1[j];
  array2[i] = array2[j];
  array1[j] = swap1;
  array2[j] = swap2;
}

/*
 * Function Name: alphabetArrayQuickSort
 *
 * Overview: Sorts an array of ULongs (largest first) and
 * applies the same changes to the array of FMons
 *
 * Detail: Using a QuickSort algorithm, this function
 * sorts an array of ULongs. The 3rd and 4th arguments
 * are used to facilitate the recursive behaviour of
 * the function -- the function should initially be called
 * as alphabetArrayQuickSort( A, B, 0, |A|-1 ).
 * It is assumed that |A| = |B| and the changes made to A
 * during the algorithm are also applied to B.
```

```
 *
 * Reference: "The C Programming Language"
 * by Brian W. Kernighan and Dennis M. Ritchie
 * (Second Edition, 1988) Page 87.
 *
 */
void
alphabetArrayQuickSort( array1, array2, start, finish )
ULong array1[];
FMon array2[];
ULong start, finish;
{
  ULong i, last;

  if( start < finish )
  {
    alphabetArraySwap( array1, array2, start, ( start + finish )/2 ); // Move partition elem
    last = start; // to array[0]

    for( i = start+1; i <= finish; i++ ) // Partition
    {
      if( array1[start] < array1[i] )
      {
        alphabetArraySwap( array1, array2, ++last, i );
      }
    }
    alphabetArraySwap( array1, array2, start, last ); // Restore partition elem
    if( last != 0 )
    {
      if( start < last-1 ) alphabetArrayQuickSort( array1, array2, start, last-1 );
    }
    if( last+1 < finish ) alphabetArrayQuickSort( array1, array2, last+1, finish );
  }
}

/*
 * Function Name: fAlgArraySwap
 *
 * Overview: Swaps 2 elements in arrays of FAlgs and FMons
 *
 * Detail: Given an array of FAlgs and an associated array
 * of FMons, this function swaps the ith and jth elements
 * of the arrays.
 *
 */
void
fAlgArraySwap( polynomials, monomials, i, j )
FAlg polynomials[];
FMon monomials[];
ULong i, j;
{
  FAlg swapA = newFAlg();
  FMon swapM = newFMon();
```

```
  swapA = polynomials[i];
  swapM = monomials[i];
  polynomials[i] = polynomials[j];
  monomials[i] = monomials[j];
  polynomials[j] = swapA;
  monomials[j] = swapM;
}

/*
 * Function Name: fAlgArrayQuickSortDRL
 *
 * Overview: Sorts an array of FAlgs using DegRevLex (largest first)
 *
 * Detail: Using a QuickSort algorithm, this function
 * sorts an array of FAlgs by sorting on the associated array
 * of FMons which store the lead monomials of the polynomials.
 * The 3rd and 4th arguments are used to facilitate the recursive
 * behaviour of the function -- the function should initially be
 * called as fAlgArrayQuickSortDRL( A, B, 0, |A|-1 ).
 *
 * Reference: "The C Programming Language"
 * by Brian W. Kernighan and Dennis M. Ritchie
 * (Second Edition, 1988) Page 87.
 *
 */
void
fAlgArrayQuickSortDRL( polynomials, monomials, start, finish )
FAlg polynomials[];
FMon monomials[];
ULong start, finish;
{
  ULong i, last;

  if( start < finish )
  {
    fAlgArraySwap( polynomials, monomials, start, ( start + finish )/2 ); // Move partition elem
    last = start; // to array[0]

    for( i = start+1; i <= finish; i++ ) // Partition
    {
      if( fMonDegRevLex( monomials[start], monomials[i] ) == (Bool) 1 )
      {
        fAlgArraySwap( polynomials, monomials, ++last, i );
      }
    }
    fAlgArraySwap( polynomials, monomials, start, last ); // Restore partition elem
    if( last != 0 )
    {
      if( start < last-1 ) fAlgArrayQuickSortDRL( polynomials, monomials, start, last-1 );
    }
    if( last+1 < finish ) fAlgArrayQuickSortDRL( polynomials, monomials, last+1, finish );
  }
```

```
}

/*
 * Function Name: fAlgArrayQuickSortOrd
 *
 * Overview: Sorts an array of FAlgs using theOrdFun (largest first)
 *
 * Detail: Using a QuickSort algorithm, this function
 * sorts an array of FAlgs by sorting on the associated array
 * of FMons which store the lead monomials of the polynomials.
 * The 3rd and 4th arguments are used to facilitate the recursive
 * behaviour of the function -- the function should initially be
 * called as fAlgArrayQuickSortOrd( A, B, 0, |A|-1 ).
 *
 * Reference: "The C Programming Language"
 * by Brian W. Kernighan and Dennis M. Ritchie
 * (Second Edition, 1988) Page 87.
 *
 */
void
fAlgArrayQuickSortOrd( polynomials, monomials, start, finish )
FAlg polynomials[];
FMon monomials[];
ULong start, finish;
{
  ULong i, last;

  if( start < finish )
  {
    fAlgArraySwap( polynomials, monomials, start, ( start + finish )/2 ); // Move partition elem
    last = start; // to array[0]

    for( i = start+1; i <= finish; i++ ) // Partition
    {
      if( theOrdFun( monomials[start], monomials[i] ) == (Bool) 1 )
      {
        fAlgArraySwap( polynomials, monomials, ++last, i );
      }
    }
    fAlgArraySwap( polynomials, monomials, start, last ); // Restore partition elem
    if( last != 0 )
    {
      if( start < last-1 ) fAlgArrayQuickSortOrd( polynomials, monomials, start, last-1 );
    }
    if( last+1 < finish ) fAlgArrayQuickSortOrd( polynomials, monomials, last+1, finish );
  }
}

/*
 * Function Name: fAlgListSort
 *
 * Overview: Sorts an FAlgList (largest first)
 *
```

```
 * Detail: This function sorts an FAlgList by
 * converting the list to an array, sorting the array
 * with a QuickSort algorithm, and converting
 * the array back to an FAlgList which is then returned.
 *
 */
FAlgList
fAlgListSort( L, type )
FAlgList L;
int type;
{
  FAlgList back = fAlgListNul;
  ULong length = fAlgListLength( L ), i;
  FAlg polynomials[length];
  FMon monomials[length];

  // Check for empty list or singleton list
  if( ( !L ) || ( length == 1 ) ) return L;

  // Transfer elements into array
  for( i = 0; i < length; i++ )
  {
    polynomials[i] = L -> first;
    monomials[i] = fAlgLeadMonom( L -> first );
    L = L -> rest;
  }

  // Sort the array (smallest -> largest)
  if( type == 1 ) // Sort by DegRevLex
    fAlgArrayQuickSortDRL( polynomials, monomials, 0, length-1 );
  else // Sort by theOrdFun
    fAlgArrayQuickSortOrd( polynomials, monomials, 0, length-1 );

  // Transfer elements back *in reverse* onto an FAlgList
  for( i = length; i >= 1; i-- )
  {
    back = fAlgListPush( polynomials[i-1], back );
  }

  // Return the sorted list
  return back;
}

/*
 * Function Name: multiplicativeArraySwap
 *
 * Overview: Swaps 2 elements in arrays of FMons, ULongs and ULongs
 *
 * Detail: Given an array of FMons and two associated arrays
 * of ULongs, this function swaps the ith and jth elements
 * of the arrays.
 *
 */
```

```
void
multiplicativeArraySwap( monomials, lengths, positions, i, j )
FMon monomials[];
ULong lengths[], positions[], i, j;
{
  FMon swapM = newFMon();
  ULong swapU1, swapU2;

  swapM = monomials[i];
  swapU1 = lengths[i];
  swapU2 = positions[i];
  monomials[i] = monomials[j];
  lengths[i] = lengths[j];
  positions[i] = positions[j];
  monomials[j] = swapM;
  lengths[j] = swapU1;
  positions[j] = swapU2;
}

/*
 * Function Name: multiplicativeQuickSort
 *
 * Overview: Sorts input data to OverlapDiv w.r.t. DegRevLex (largest first)
 *
 * Detail: Using a QuickSort algorithm, this function
 * sorts an array of FMons w.r.t. DegRevLex and applies the same
 * changes to two associated arrays of ULongs.
 * The 4th and 5th arguments are used to facilitate the recursive
 * behaviour of the function -- the function should initially be
 * called as multiplicativeQuickSort( A, B, C, 0, |A|-1 ).
 *
 * Reference: "The C Programming Language"
 * by Brian W. Kernighan and Dennis M. Ritchie
 * (Second Edition, 1988) Page 87.
 *
 */
void
multiplicativeQuickSort( monomials, lengths, positions, start, finish )
FMon monomials[];
ULong lengths[], positions[], start, finish;
{
  ULong i, last;

  if( start < finish )
  {
    // Move partition elem to array[0]
    multiplicativeArraySwap( monomials, lengths, positions, start, ( start + finish )/2 );
    last = start;

    for( i = start+1; i <= finish; i++ ) // Partition
    {
      if( fMonDegRevLex( monomials[start], monomials[i] ) == (Bool) 1 )
      {
```

```
        multiplicativeArraySwap( monomials, lengths, positions, ++last, i );
      }
    }
    multiplicativeArraySwap( monomials, lengths, positions, start, last ); // Restore partition elem
    if( last != 0 )
    {
      if( start < last-1 ) multiplicativeQuickSort( monomials, lengths, positions, start, last-1 );
    }
    if( last+1 < finish ) multiplicativeQuickSort( monomials, lengths, positions, last+1, finish );
  }
}

/*
 * ========================
 * Insertion Sort Functions
 * ========================
 */

/*
 * Function Name: fAlgListDegRevLexPush
 *
 * Overview: Insert into list according to DegRevLex
 *
 * Detail: This functions inserts the polynomial _poly_
 * into the FAlgList _input_ so that the list remains
 * sorted by DegRevLex (largest first).
 *
 */
FAlgList
fAlgListDegRevLexPush( poly, input )
FAlg poly;
FAlgList input;
{
  FAlgList output = fAlgListNul; // Initialise the return list
  FMon lead = fAlgLeadMonom( poly );

  if( !input ) // If there is nothing in the input list
  {
    // Return a singleton list
    return fAlgListSingle( poly );
  }
  else
  {
    // While the next element in the list is larger than _lead_
    while( ( fAlgListLength( input ) > 0 )
           && ( fMonDegRevLex( lead, fAlgLeadMonom( input -> first ) ) == (Bool) 1 ) )
    {
      // Push the list element onto the output list
      output = fAlgListPush( input -> first, output );
      input = input -> rest; // Advance the list
    }
    // Now push the new element onto the list
    output = fAlgListPush( poly, output );
```

```
    // Reverse the output list (it was constructed in reverse)
    output = fAlgListFXRev( output );
    // If there is anything left in the input list, tag it onto the output list
    if( input ) output = fAlgListAppend( output, input );

    return output;
  }
}

/*
 * Function Name: fAlgListDegRevLexPushPosition
 *
 * Overview: As above, but also returns the insertion position
 *
 * Detail: This functions inserts the polynomial _poly_
 * into the FAlgList _input_ so that the list remains
 * sorted by DegRevLex (largest first). The position in
 * which the insertion took place is placed in the
 * variable _pos_.
 *
 */
FAlgList
fAlgListDegRevLexPushPosition( poly, input, pos )
FAlg poly;
FAlgList input;
ULong *pos;
{
  FAlgList output = fAlgListNul; // Initialise the return list
  FMon lead = fAlgLeadMonom( poly );
  ULong position = 1;

  if( !input ) // If there is nothing in the input list
  {
    *pos = 1; // Inserted into the first position
    // Return a singleton list
    return fAlgListSingle( poly );
  }
  else
  {
    // While the next element in the list is larger than _lead_
    while( ( fAlgListLength( input ) > 0 )
           && ( fMonDegRevLex( lead, fAlgLeadMonom( input -> first ) ) == (Bool) 1 ) )
    {
      // Push the list element onto the output list
      output = fAlgListPush( input -> first, output );
      input = input -> rest; // Advance the list
      position++; // Increment the insertion position
    }
    // We now know the insertion position
    *pos = position;
    // Push the new element onto the list
    output = fAlgListPush( poly, output );
    // Reverse the output list (it was constructed in reverse)
```

```
    output = fAlgListFXRev( output );
    // If there is anything left in the input list, tag it onto the output list
    if( input ) output = fAlgListAppend( output, input );

    return output;
  }
}

/*
 * Function Name: fAlgListNormalPush
 *
 * Overview: Insert into list according to the current monomial ordering
 *
 * Detail: This functions inserts the polynomial _poly_
 * into the FAlgList _input_ so that the list remains
 * sorted by the current monomial ordering (largest first).
 *
 */
FAlgList
fAlgListNormalPush( poly, input )
FAlg poly;
FAlgList input;
{
  FAlgList output = fAlgListNul; // Initialise the return list
  FMon lead = fAlgLeadMonom( poly );

  if( !input ) // If there is nothing in the input list
  {
    // Return a singleton list
    return fAlgListSingle( poly );
  }
  else
  {
    // While the next element in the list is larger than _lead_
    while( ( fAlgListLength( input ) > 0 )
           && ( theOrdFun( lead, fAlgLeadMonom( input -> first ) ) == (Bool) 1 ) )
    {
      // Push the list element onto the output list
      output = fAlgListPush( input -> first, output );
      input = input -> rest; // Advance the list
    }
    // Now push the new element onto the list
    output = fAlgListPush( poly, output );
    // Reverse the output list (it was constructed in reverse)
    output = fAlgListFXRev( output );
    // If there is anything left in the input list, tag it onto the output list
    if( input ) output = fAlgListAppend( output, input );

    return output;
  }
}

/*
```

```
 * Function Name: fAlgListNormalPushPosition
 *
 * Overview: As above, but also returns the insertion position
 *
 * Detail: This functions inserts the polynomial _poly_
 * into the FAlgList _input_ so that the list remains
 * sorted by the current monomial ordering (largest first).
 * The position in which the insertion took place is placed
 * in the variable _pos_.
 *
 */
FAlgList
fAlgListNormalPushPosition( poly, input, pos )
FAlg poly;
FAlgList input;
ULong *pos;
{
  FAlgList output = fAlgListNul; // Initialise the return list
  FMon lead = fAlgLeadMonom( poly );
  ULong position = 1;

  if( !input ) // If there is nothing in the input list
  {
    *pos = 1; // Inserted into the first position
    // Return a singleton list
    return fAlgListSingle( poly );
  }
  else
  {
    // While the next element in the list is larger than _lead_
    while( ( fAlgListLength( input ) > 0 )
           && ( theOrdFun( lead, fAlgLeadMonom( input -> first ) ) == (Bool) 1 ) )
    {
      // Push the list element onto the output list
      output = fAlgListPush( input -> first, output );
      input = input -> rest; // Advance the list
      position++; // Increment the insertion position
    }
    // We now know the insertion position
    *pos = position;
    // Push the new element onto the list
    output = fAlgListPush( poly, output );
    // Reverse the output list (it was constructed in reverse)
    output = fAlgListFXRev( output );
    // If there is anything left in the input list, tag it onto the output list
    if( input ) output = fAlgListAppend( output, input );

    return output;
  }
}

/*
 * ==================
```

```
 * Deletion Functions
 * ==================
 */

/*
 * Function Name: fMonListRemoveNumber
 *
 * Overview: Removes the (1st arg)-th element from the list
 *
 * Detail: Given an FMonList _list_, this function removes
 * from _list_ the element in position _number_.
 *
 */
FMonList
fMonListRemoveNumber( number, list )
ULong number;
FMonList list;
{
  FMonList output = fMonListNul;
  ULong i;

  for( i = 1; i < number; i++ )
  {
    // Push the first (number-1) elements onto the list
    output = fMonListPush( list -> first, output );
    list = list -> rest;
  }

  // Delete the number-th element by skipping past it
  list = list -> rest;

  // Push the remaining elements onto the list
  while( list )
  {
    output = fMonListPush( list -> first, output );
    list = list -> rest;
  }

  // Return the reversed list (it was constructed in reverse)
  return fMonListFXRev( output );
}

/*
 * Function Name: fMonPairListRemoveNumber
 *
 * Overview: Removes the (1st arg)-th element from the list
 *
 * Detail: Given an FMonPairList _list_, this function removes
 * from _list_ the element in position _number_.
 *
 */
FMonPairList
fMonPairListRemoveNumber( number, list )
```

```
ULong number;
FMonPairList list;
{
  FMonPairList output = fMonPairListNul;
  ULong i;

  for( i = 1; i < number; i++ )
  {
    // Push the first (number−1) elements onto the list
    output = fMonPairListPush( list −> lft, list −> rt, output );
    list = list −> rest;
  }

  // Delete the number−th element by skipping past it
  list = list −> rest;

  // Push the remaining elements onto the list
  while( list )
  {
    output = fMonPairListPush( list −> lft, list −> rt, output );
    list = list −> rest;
  }

  // Return the reversed list (it was constructed in reverse)
  return fMonPairListFXRev( output );
}

/*
 * Function Name: fAlgListRemoveNumber
 *
 * Overview: Removes the (1st arg)−th element from the list
 *
 * Detail: Given an FAlgList _list_, this function removes
 * from _list_ the element in position _number_.
 *
 */
FAlgList
fAlgListRemoveNumber( number, list )
ULong number;
FAlgList list;
{
  FAlgList output = fAlgListNul;
  ULong i;

  for( i = 1; i < number; i++ )
  {
    // Push the first (number−1) elements onto the list
    output = fAlgListPush( list −> first, output );
    list = list −> rest;
  }

  // Delete the number−th element by skipping past it
  list = list −> rest;
```

```

  // Push the remaining elements onto the list
  while( list )
  {
    output = fAlgListPush( list -> first, output );
    list = list -> rest;
  }

  // Return the reversed list (it was constructed in reverse)
  return fAlgListFXRev( output );
}

/*
 * =====================
 * Normalising Functions
 * =====================
 */

/*
 * Function Name: fAlgListRemoveFractions
 *
 * Overview: Removes any fractions found in the FAlgList by scalar multiplication
 *
 * Detail: Given a list of polynomials, this function analyses
 * each polynomial in turn, multiplying a polynomial by an
 * appropriate integer if a fractional coefficient is
 * found for any term in the polynomial. For example, if one
 * polynomial in the list is (2/3)xy + (1/5)x + 2y,
 * then the polynomial is multiplied by 3*5 = 15 to remove
 * the fractional coefficients, and the output polynomial
 * is therefore 10xy + 3x + 30y.
 *
 */
FAlgList
fAlgListRemoveFractions( input )
FAlgList input;
{
  FAlgList output = fAlgListNul;
  FAlg p, LTp, new;
  Integer denominator;

  while( input ) // For each polynomial in the list
  {
    p = input -> first; // Extract a polynomial
    input = input -> rest; // Advance the list

    new = fAlgZero(); // Initialise the new polynomial
    while( p ) // For each term of the polynomial p
    {
      LTp = fAlgLeadTerm( p ); // Extract the lead term
      p = fAlgReductum( p ); // Advance the polynomial

      denominator = fAlgLeadCoef( LTp ) -> den; // Extract the denominator
```

```
      if( zIsOne( denominator ) == 0 ) // If the denominator is not 1
      {
        // Multiply the whole polynomial by the denominator
        if( p ) p = fAlgZScaTimes( denominator, p ); // Still to be looked at
        LTp = fAlgZScaTimes( denominator, LTp ); // Looking at
        new = fAlgZScaTimes( denominator, new ); // Looked at
      }
      new = fAlgPlus( new, LTp ); // Add the term to the output polynomial
    }
    output = fAlgListPush( new, output ); // Add the new polynomial to the output list
  }

  // The new list was read in reverse so we must reverse it before returning it
  return fAlgListFXRev( output );
}

/*
 * ===========
 * End of File
 * ===========
 */
```

B.2.10 ncinv_functions.h

```
/*
 * File: ncinv_functions.h
 * Author: Gareth Evans
 * Last Modified: 6th July 2005
 */

// Initialise file definition
# ifndef NCINV_FUNCTIONS_HDR
# define NCINV_FUNCTIONS_HDR

// Include MSSRC Libraries
# include <fralg.h>

//
// External Variables Required
//

extern ULong nOfProlongations, // Stores the number of prolongations calculated
             nRed; // Stores how many reductions have been performed
extern int degRestrict, // Determines whether of not prolongations are restricted by degree
           EType, // Stores the type of Overlap Division
           IType, // Stores the involutive division used
           nOfGenerators, // Holds the number of generators
           pl, // Holds the "Print Level"
           SType, // Determines how the basis is sorted
           MType; // Determines involutive division method

//
```

```
// Functions Defined in ncinv_functions.c
//

//
// Overlap Functions
//

// Returns the union of (non-)multiplicative variables (1st arg) and a generator (2nd arg)
FMon multiplicativeUnion( FMon, FMon );
// Does the generator (1st arg) appear in the list of multiplicative variables (2nd arg)?
int fMonIsMultiplicative( FMon, FMon );
// Does the 1st arg appear as a subword in the 2nd arg (yes (1)/no (0))
int fMonIsSubword( FMon, FMon );
// Is the 1st arg a subword of the 2nd arg; if so, return start pos in 2nd arg
ULong fMonSubwordOf( FMon, FMon, ULong );

// Returns size of smallest overlap of type (suffix of 1st arg = prefix of 2nd arg)
ULong fMonPrefixOf( FMon, FMon, ULong, ULong );
// Returns size of smallest overlap of type (prefix of 1st arg = suffix of 2nd arg)
ULong fMonSuffixOf( FMon, FMon, ULong, ULong );

//
// Multiplicative Variables Functions
//

// Returns no ('empty') multiplicative variables
void EMultVars( FMon, ULong *, ULong * );
// All variables left mult., no variables right mult.
void LMultVars( FMon, ULong *, ULong * );
// All variables right mult., no variables left mult.
void RMultVars( FMon, ULong *, ULong * );
// Returns local overlap-based multiplicative variables
FMonPairList OverlapDiv( FAlgList );

//
// Polynomial Reduction and Basis Completion Functions
//

// Reduces 1st arg w.r.t. 2nd arg (list) and 3rd arg (vars)
FAlg IPolyReduce( FAlg, FAlgList, FMonPairList );
// Autoreduces an FAlgList recursively until no more reductions are possible
FAlgList IAutoreduceFull( FAlgList );
// Implements Seiler's original algorithm for computing locally involutive bases
FAlgList Seiler( FAlgList );
// Implements Gerdt's advanced algorithm for computing locally involutive bases
FAlgList Gerdt( FAlgList );

#endif // NCINV_FUNCTIONS_HDR
```

B.2.11 ncinv_functions.c

```
/*
```

```
 * File: ncinv_functions.c
 * Author: Gareth Evans
 * Last Modified: 10th August 2005
 */

/*
 * =======================================
 * Global Variables for ncinv_functions.c
 * =======================================
 */

int headReduce = 0; // Controls type of polynomial reduction
ULong d, // Stores the bound on the restriction of prolongations
      twod; // Stores 2*d for efficiency

/*
 * =================
 * Overlap Functions
 * =================
 */

/*
 * Function Name: multiplicativeUnion
 *
 * Overview: Returns the union of (non-)multiplicative variables
 * (1st arg) and a generator (2nd arg)
 *
 * Detail: This function inserts a generator into a monomial representing
 * (non-)multiplicative variables so that the ASCII ordering of the
 * monomial is preserved. For example, if _a_ = A*B*C*E*F and _b_ = D,
 * then the output monomial is A*B*C*D*E*F.
 *
 */
FMon
multiplicativeUnion( a, b )
FMon a, b;
{
  FMon output = fMonOne();
  ULong test, insert = ASCIIVal( fMonLeadVar( b ) ),
        len = fMonLength( a );

  // If a is empty there is no problem - we just return b
  if( !a ) return b;
  else
  {
    // Go through each generator in a
    while( len > 0 )
    {
      len--;
      // Obtain the numerical value of the first generator
      test = ASCIIVal( fMonLeadVar( a ) );

      if( test < insert ) // We must skip past this generator
```

```
        output = fMonTimes( output, fMonPrefix( a, 1 ) );
      else if( test == insert ) // b is already in a so we just return the _original_ a
        return fMonTimes( output, a );
      else // We insert b in this position and tag on the remainder
        return fMonTimes( output, fMonTimes( b, a ) );

      // Get ready to look at the next generator
      a = fMonTailFac( a );
    }
  }

  // Deal with the case "insert > {everything in a}"
  return fMonTimes( output, b );
}

/*
 * Function Name: fMonIsMultiplicative
 *
 * Overview: Does the generator _a_ appear in the list of multiplicative variables _b_?
 *
 * Detail: Given a generator _a_, this function tests to see whether
 * _a_ appears in a list of multiplicative variables _b_.
 *
 */
int
fMonIsMultiplicative( a, b )
FMon a, b;
{
  ULong lenb = fMonLength( b ), i;

  // For each possible overlap
  for( i = 1; i <= lenb; i++ )
  {
    if( fMonEqual( a, fMonSubWordLen( b, i, 1 ) ) == (Bool) 1 )
      return 1; // Match found
  }

  return 0; // No match found
}

/*
 * Function Name: fMonIsSubword
 *
 * Overview: Does _a_ appear as a subword in _b_ (yes (1)/no (0))
 *
 * Detail: This function answers the question "Is _a_ a subword of _b_?"
 * The function returns 1 if _a_ is a subword of _b_ and 0 otherwise.
 *
 */
int
fMonIsSubword( a, b )
FMon a, b;
{
```

```
  ULong lena = fMonLength( a ), lenb = fMonLength( b ), i;

  // For each possible overlap
  for( i = 1; i <= lenb−lena+1; i++ )
  {
    if( fMonEqual( a, fMonSubWordLen( b, i, lena ) ) == (Bool) 1 )
      return 1; // Overlap found
  }

  return 0; // No overlap found
}

/*
 * Function Name: fMonSubwordOf
 *
 * Overview: Is the 1st arg a subword of the 2nd arg; if so, return start pos in 2nd arg
 *
 * Detail: This function can answer the question "Is _small_ a subword of _large_?"
 * The function returns i if _small_ is a subword of _large_,
 * where i is the position in _large_ of the first subword found,
 * and returns 0 if no overlap exists. We start looking for subwords starting
 * at position _start_ in _large_ and finish looking for subwords when
 * all possibilities have been exhausted (we work left−to−right). It follows
 * that to test all possibilities the 3rd argument should be 1, but note that
 * you should use the above function (fMonIsSubword) if you only want to know
 * if a monomial is a subword of another monomial and are not fussed
 * where the overlap takes place.
 *
 */
ULong
fMonSubwordOf( small, large, start )
FMon small, large;
ULong start;
{
  ULong i = start, sLen = fMonLength( small ), lLen = fMonLength( large );

  // While there are more subwords to test for
  while( i <= lLen−sLen+1 )
  {
    // If small is equal to a subword of large
    if( fMonEqual( small, fMonSubWordLen( large, i, sLen ) ) == (Bool) 1 )
    {
      return i; // Subword found
    }
    i++;
  }
  return 0; // No subwords found
}

/*
 * Function Name: fMonPrefixOf
 *
 * Overview: Returns size of smallest overlap of type (suffix of 1st arg = prefix of 2nd arg)
```

```
 *
 * Detail: This function can answer the question "Is _left_ a prefix of _right_?"
 * The function returns i if a suffix of _left_ is equal to a prefix of _right_,
 * where i is the length of the smallest overlap, and returns 0 if no overlap exists.
 * The lengths of the overlaps we look at are controlled by the 3rd and 4th
 * arguments - we start by looking at the overlap of size _start_ and finish
 * by looking at the overlap of size _limit_. It is the user's responsibility
 * to ensure that these bounds are correct - no checks are made by the function.
 * To test all possibilities, the 3rd argument should be 1 and the fourth
 * argument should be min( |left|, |right| ) - 1.
 *
 */
ULong
fMonPrefixOf( left, right, start, limit )
FMon left, right;
ULong start, limit;
{
  ULong i = start;

  while( i <= limit ) // For each overlap
  {
    if( fMonEqual( fMonSuffix( left, i ), fMonPrefix( right, i ) ) == (Bool) 1 )
    {
      return i; // Prefix found
    }
    i++;
  }
  return 0; // No prefixes found
}

/*
 * Function Name: fMonSuffixOf
 *
 * Overview: Returns size of smallest overlap of type (prefix of 1st arg = suffix of 2nd arg)
 *
 * Detail: This function can answer the question "Is _left_ a suffix of _right_?"
 * The function returns i if a prefix of _left_ is equal to a suffix of _right_,
 * where i is the length of the smallest overlap, and returns 0 if no overlap exists.
 * The lengths of the overlaps we look at are controlled by the 3rd and 4th
 * arguments - we start by looking at the overlap of size _start_ and finish
 * by looking at the overlap of size _limit_. It is the user's responsibility
 * to ensure that these bounds are correct - no checks are made by the function.
 * To test all possibilities, the 3rd argument should be 1 and the fourth
 * argument should be min( |left|, |right| ) - 1.
 *
 */
ULong
fMonSuffixOf( left, right, start, limit )
FMon left, right;
ULong start, limit;
{
  ULong i = start;

```

```
  while( i <= limit ) // For each overlap
  {
    if( fMonEqual( fMonPrefix( left, i ), fMonSuffix( right, i ) ) == (Bool) 1 )
    {
      return i; // Suffix found
    }
    i++;
  }
  return 0; // No suffixes found
}

/*
 * ===================================
 * Multiplicative Variables Functions
 * ===================================
 */

/*
 * Function Name: EMultVars
 *
 * Overview: Returns no ('empty') multiplicative variables
 *
 * Detail: Given a monomial, this function assigns
 * no multiplicative variables.
 *
 * External Variables Required: int nOfGenerators;
 *
 */
void
EMultVars( mon, max, min )
FMon mon;
ULong *max, *min;
{
   // Nothing is right multiplicative
   *max = (ULong)nOfGenerators + 1;
   // Nothing is left multiplicative
   *min = 0;
}

/*
 * Function Name: LMultVars
 *
 * Overview: All variables left mult., no variables right mult.
 *
 * Detail: Given a monomial, this function assigns
 * all variables to be left multiplicative and all
 * variables to be right nonmultiplicative.
 *
 * External Variables Required: int nOfGenerators;
 *
 */
void
LMultVars( mon, max, min )
```

```
FMon mon;
ULong *max, *min;
{
    // Nothing is right multiplicative
    *max = (ULong)nOfGenerators + 1;
    // Everything is left multiplicative
    *min = (ULong)nOfGenerators + 1;
}

/*
 * Function Name: RMultVars
 *
 * Overview: All variables right mult., no variables left mult.
 *
 * Detail: Given a monomial, this function assigns
 * all variables to be right multiplicative and all
 * variables to be left nonmultiplicative.
 *
 * External Variables Required: int nOfGenerators;
 *
 */
void
RMultVars( mon, max, min )
FMon mon;
ULong *max, *min;
{
    // Everything is right multiplicative
    *max = 0;
    // Nothing is left multiplicative
    *min = 0;
}

/*
 * Function Name: OverlapDiv
 *
 * Overview: Returns local overlap-based multiplicative variables
 *
 * Detail: This function implements various algorithms
 * described in the thesis "Noncommutative Involutive Bases"
 * for finding left and right multiplicative variables
 * for a set of polynomials based on the overlaps
 * between the leading monomials of the polynomials.
 *
 * External Variables Required: int EType, IType, nOfGenerators, pl, SType;
 *
 */
FMonPairList
OverlapDiv( list )
FAlgList list;
{
  FMonPairList output = fMonPairListNul;
  FMon generator;
  ULong listLen = fAlgListLength( list ),
```

```
      monLength[listLen], tracking[listLen],
      i, j, first, limit, result, len,
      letterVal1, letterVal2;
FMon monomials[listLen], monExcl,
     leftMult[listLen], rightMult[listLen];
short grid[listLen][(ULong)nOfGenerators * 2],
      thresholdBroken, excludeL, excludeR;

// Give some initial information
if( pl > 3 )
{
  printf("OverlapDiv's Input = \n");
  fAlgListDisplay( list );
}

if( !list ) return output;

// Set up arrays
i = 0;
while( list ) // For each polynomial
{
  monomials[i] = fAlgLeadMonom( list -> first ); // Extract lead monomial
  monLength[i] = fMonLength( monomials[i] ); // Find monomial length
  leftMult[i] = fMonOne(); // Initialise left multiplicative variables
  rightMult[i] = fMonOne(); // Initialise right multiplicative variables
  for( j = 0; j < (ULong) nOfGenerators*2; j++ )
  {
    /*
     * Fill the multiplicative grid with 1's,
     * where the columns of the grid are
     * gen_1^L, gen_1^R, gen_2^L, gen_2^R, ..., gen_{nOfGenerators}^R
     * and the rows of the grid are
     * monomials[0], monomials[1], ..., monomials[listLen].
     */
    grid[i][j] = 1;
  }
  // If SType > 1 we need to sort the basis first, keeping track of the changes made
  if( SType > 1 ) tracking[i] = i;
  i++;
  list = list -> rest; // Advance the list
}

if( pl > 7 ) printf("Arrays Set Up (size of input basis = %u)\n", listLen);

// If SType > 1 and there is more than one polynomial in the basis,
// we need to sort the basis w.r.t. DegRevLex (Greatest first) in order
// to be able to apply the algorithm.
if( ( SType > 1 ) && ( listLen > 1 ) )
{
  multiplicativeQuickSort( monomials, monLength, tracking, 0, listLen - 1 );

  if( pl > 6 )
  {
```

```
    printf("Sorted Input = \n");
    for( i = 0; i < listLen; i++ ) printf("%s\n", fMonToStr( monomials[i] ) );
  }
}

/*
 * Now exclude multiplicative variables based on overlaps
 */

// For each monomial
for( i = 0; i < listLen; i++ )
{
  thresholdBroken = 0;
  for( j = i; j < listLen; j++ ) // For each monomial less than or equal to monomial i in DRL
  {
    /*
     * To look for subwords, the length of monomial j has to
     * be less than the length of monomial i. We use the variable
     * thresholdBroken to store whether monomials of length less
     * than the length of monomial i have been encountered yet,
     * and obviously we must have j > i for this to be the case.
     */
    if( ( j > i ) && ( thresholdBroken == 0 ) )
    {
      if( monLength[j] < monLength[i] )
        thresholdBroken = 1; // if deg(j) < deg(i) we can now start to consider subwords
    }
    if( ( thresholdBroken == 1 ) && ( EType != 5 ) ) // Stage 1: Look for subwords
    {
      first = 1;
      // There are monLength[i] - monLength[j] + 1 test subwords in all
      limit = monLength[i] - monLength[j] + 1;
      // Test whether monomial j is a subword of monomial i, starting with the first subword
      result = fMonSubwordOf( monomials[j], monomials[i], first );
      if( pl > 8 ) printf("fMonSubwordOf( %s, %s, %u ) = %u\n", fMonToStr( monomials[j] ),
                          fMonToStr( monomials[i] ), first, result );

      while( result != 0 ) // While there are subwords to be processed
      {
        if( IType == 1 ) // Left Overlap Division
        {
          if( result < limit )
          {
            if( ( EType < 4 ) || ( ( EType == 4 ) && ( result == 1 ) ) )
            {
              /*
               * Exclude right multiplicative variable - overlap of type 'B' or 'C'
               * ------------- monomial[i]
               * ---------x monomial[j] (space on the right)
               * Note: the above diagram (and the following diagrams) may
               * not appear correctly in Appendix B due to using flexible columns.
               * The correct diagrams (referenced by the letters 'A' to 'D' can
               * be found in the README file in Appendix B.
```

```
             */
            generator = fMonSubWordLen( monomials[i], result + monLength[j], 1 );
            letterVal1 = ASCIIVal( fMonLeadVar( generator ) ) - 1;
            grid[j][2*letterVal1+1] = 0; // Set right non multiplicative
          }
        }
        else if( EType == 3 )
        {
          /*
           * Exclude left multiplicative variable - overlap of type 'D'
           * ------------- monomial[i]
           * x-------- monomial[j] (no space on the right)
           */
          generator = fMonSubWordLen( monomials[i], result-1, 1 );
          letterVal1 = ASCIIVal( fMonLeadVar( generator ) ) - 1;
          grid[j][2*letterVal1] = 0; // Set left non multiplicative
        }
      }
      else // Right Overlap Division
      {
        if( result > 1 )
        {
          if( ( EType < 4 ) || ( ( EType == 4 ) && ( result == limit ) ) )
          {
            /*
             * Exclude left multiplicative variable - overlap of type 'B' or 'C'
             * ------------- monomial[i]
             * x------ monomial[j] (space on the left)
             */
            generator = fMonSubWordLen( monomials[i], result-1, 1 );
            letterVal1 = ASCIIVal( fMonLeadVar( generator ) ) - 1;
            grid[j][2*letterVal1] = 0; // Set left non multiplicative
          }
        }
        else if( EType == 3 )
        {
          /*
           * Exclude right multiplicative variable - overlap of type 'D'
           * ------------- monomial[i]
           * ---------x monomial[j] (no space on the left)
           */
          generator = fMonSubWordLen( monomials[i], result + monLength[j], 1 );
          letterVal1 = ASCIIVal( fMonLeadVar( generator ) ) - 1;
          grid[j][2*letterVal1+1] = 0; // Set right non multiplicative
        }
      }

      // We will now look for the next available subword
      first = result + 1;
      if( first <= limit ) // If the limit has not been exceeded
      {
        result = fMonSubwordOf( monomials[j], monomials[i], first ); // Look for more subwords
        if( pl > 8 ) printf("fMonSubwordOf( %s, %s, %u ) = %u\n", fMonToStr( monomials[j] ),
```

```
                    fMonToStr( monomials[i] ), first, result );
      }
      else // Otherwise exit from the loop
        result = 0;
    }
  }

  // Stage 2: Look for prefixes
  first = 1;
  // There are monLength[j] − 1 test prefixes in all
  limit = monLength[j] − 1;
  // Test whether a suffix of monomial j is a prefix of monomial i, starting with the prefix of length 1
  result = fMonPrefixOf( monomials[j], monomials[i], first, limit );
  if( pl > 8 ) printf("fMonPrefixOf( %s, %s, %u, %u ) = %u\n", fMonToStr( monomials[j] ),
                 fMonToStr( monomials[i] ), first, limit, result );

  while( result != 0 ) // While there are prefixes to be processed
  {
    /*
     * Possibly exclude right multiplicative variable − overlap of type 'A'
     * 1−−−−−−−−−−−−− monomial[i]
     * −−−−−−−−−2 monomial[j]
     */
    generator = fMonSubWordLen( monomials[j], monLength[j] − result, 1 );
    letterVal1 = ASCIIVal( fMonLeadVar( generator ) ) − 1;
    generator = fMonSubWordLen( monomials[i], result + 1, 1 );
    letterVal2 = ASCIIVal( fMonLeadVar( generator ) ) − 1;

    if( IType == 1 ) // Left Overlap Division
    {
      if( EType != 3 ) // Assign right nonmultiplicative
      {
        grid[j][2*letterVal2+1] = 0; // Set j right non multiplicative for '2'
      }
      else // Assign nonmultiplicative only if both currently multiplicative
      {
        // If monomial i is left multiplicative for '1' and j right multiplicative for '2'
        if( grid[i][2*letterVal1] + grid[j][2*letterVal2+1] == 2 )
          grid[j][2*letterVal2+1] = 0; // Set j right non multiplicative for '2'
      }
    }
    else // Right Overlap Division
    {
      if( EType != 3 ) // Assign left nonmultiplicative
      {
        grid[i][2*letterVal1] = 0; // Set i left non multiplicative for '1'
      }
      else // Assign nonmultiplicative only if both currently multiplicative
      {
        // If monomial i is left multiplicative for '1' and j right multiplicative for '2'
        if( grid[i][2*letterVal1] + grid[j][2*letterVal2+1] == 2 )
          grid[i][2*letterVal1] = 0; // Set i left non multiplicative for '1'
      }
```

```
      }

      // We will now look for the next available suffix
      first = result + 1;
      if( first <= limit ) // If the limit has not been exceeded
      {
        result = fMonPrefixOf( monomials[j], monomials[i], first, limit ); // Look for more prefixes
        if( pl > 8 ) printf("fMonPrefixOf( %s, %s, %u, %u ) = %u\n", fMonToStr( monomials[j] ),
                              fMonToStr( monomials[i] ), first, limit, result );
      }
      else // Otherwise exit from the loop
        result = 0;
    }

    // Stage 3: Look for suffixes
    first = 1;
    // There are monLength[j] - 1 test suffixes in all
    limit = monLength[j] - 1;
    // Test whether a prefix of monomial j is a suffix of monomial i, starting with the suffix of length 1
    result = fMonSuffixOf( monomials[j], monomials[i], first, limit );
    if( pl > 8 ) printf("fMonSuffixOf( %s, %s, %u, %u ) = %u\n", fMonToStr( monomials[j] ),
                          fMonToStr( monomials[i] ), first, limit, result );

    while( result != 0 ) // While there are suffixes to be processed
    {
      /*
       * Possibly exclude left multiplicative variable - overlap of type 'A'
       * -------------1 monomial[i]
       * 2--------- monomial[j]
       */
      generator = fMonSubWordLen( monomials[j], result + 1, 1 );
      letterVal1 = ASCIIVal( fMonLeadVar( generator ) ) - 1;
      generator = fMonSubWordLen( monomials[i], monLength[i] - result, 1 );
      letterVal2 = ASCIIVal( fMonLeadVar( generator ) ) - 1;

      if( IType == 1 ) // Left Overlap Division
      {
        if( EType != 3 ) // Assign right nonmultiplicative
        {
          grid[i][2*letterVal1+1] = 0; // Set i right non multiplicative for '1'
        }
        else // Assign nonmultiplicative only if both currently multiplicative
        {
          // If monomial i is right multiplicative for '1' and j left multiplicative for '2'
          if( grid[i][2*letterVal1+1] + grid[j][2*letterVal2] == 2 )
            grid[i][2*letterVal1+1] = 0; // Set i right non multiplicative for '1'
        }
      }
      else // Right Overlap Division
      {
        if( EType != 3 ) // Assign left nonmultiplicative
        {
          grid[j][2*letterVal2] = 0; // Set j left non multiplicative for '2'
```

```
          }
          else // Assign nonmultiplicative only if both currently multiplicative
          {
            // If monomial i is right multiplicative for '1' and j left multiplicative for '2'
            if( grid[i][2*letterVal1+1] + grid[j][2*letterVal2] == 2 )
              grid[j][2*letterVal2] = 0; // Set j left non multiplicative for '2'
          }
        }

        // We will now look for the next available suffix
        first = result + 1;
        if( first <= limit ) // If the limit has not been exceeded
        {
          result = fMonSuffixOf( monomials[j], monomials[i], first, limit ); // Look for more suffixes
          if( pl > 8 ) printf("fMonSuffixOf( %s, %s, %u, %u ) = %u\n", fMonToStr( monomials[j] ),
                              fMonToStr( monomials[i] ), first, limit, result );
        }
        else // Otherwise exit from the loop
          result = 0;
      }
    }
  }

  if( EType == 2 )
  {
    // Ensure all cones are disjoint
    for( i = listLen; i > 0; i-- ) // For each monomial (working up)
    {
      for( j = listLen; j > 0; j-- ) // For each monomial
      {
        /*
         * We will now make sure that some variable in monomial[j] is
         * right (left) nonmultiplicative for monomial[i].
         */

        // Assume to begin with that the above holds
        if( IType == 1 )
        {
          first = 1; // Used to find the first variable
          excludeL = 0;
        }
        else excludeR = 0;

        monExcl = monomials[j-1]; // Extract a monomial for processing
        len = fMonLength( monExcl ); // Find the length of monExcl

        while( ( len > 0 ) && ( ( excludeL + excludeR ) != 1 ) ) // For each variable in monomial[j]
        {
          len = len - fMonLeadExp( monExcl );

          // Extract a variable
          letterVal1 = ASCIIVal( fMonLeadVar( monExcl ) ) - 1;

```

```
          if( lType == 1 )
          {
            if( first == 1 )
            {
              letterVal2 = letterVal1; // Store the first variable encountered
              first = 0; // To ensure this code only runs once
            }
          }

          if( lType == 1 ) // Left Overlap Division
          {
            // If this variable is right nonmultiplicative for monomial[i], change excludeL
            if( grid[i−1][2∗letterVal1+1] == 0 ) excludeL = 1;
          }
          else // Right Overlap Division
          {
            // If this variable is left nonmultiplicative for monomial[i], change excludeR
            if( grid[i−1][2∗letterVal1] == 0 ) excludeR = 1;
          }
          monExcl = fMonTailFac( monExcl ); // Get ready to look at the next variable
        }

        if( lType == 1 ) // Left Overlap Division
        {
          // If no variable was right nonmultiplicative for monomial[i]...
          if( excludeL == 0 )
            grid[i−1][2∗letterVal2+1] = 0; // ...set the first variable encountered to be right nonmultiplicative
        }
        else // Right Overlap Division
        {
          // If no variable was left nonmultiplicative for monomial[i]...
          if( excludeR == 0 )
            grid[i−1][2∗letterVal1] = 0; // ...set the last variable encountered to be left nonmultiplicative
        }
      }
    }
  }

  // Provide some intermediate output information
  if( pl > 6 )
  {
    printf("Multiplicative Grid:\n");
    for( i = 0; i < listLen; i++ )
    {
      printf("Monomial %u = %s:\n", i, fMonToStr( monomials[i] ) );
      for( j = 0; j < (ULong) nOfGenerators ∗ 2; j++ ) printf("%i, ", grid[i][j] );
      printf("\n");
    }
    printf("\n");
  }

  /∗
   ∗ Convert the grid to 2 arrays of FMons, where
```

```
 * each FMon stores a list of multiplicative variables
 * in increasing variable order
 */

if( SType > 1 ) // Need to sort as well
{
  // Convert the grid to monomial data
  for( i = 0; i < listLen; i++ ) // For each monomial
  {
    for( j = 0; j < (ULong) nOfGenerators; j++ ) // For each variable
    {
      if( grid[i][2*j] == 1 ) // LEFT Assigned
      {
        // Multiply on the left by a multiplicative variable
        leftMult[tracking[i]] = fMonTimes( leftMult[tracking[i]], ASCIIMon( j+1 ) );
      }
      if( grid[i][2*j+1] == 1 ) // RIGHT Assigned
      {
        // Multiply on the left by a multiplicative variable
        rightMult[tracking[i]] = fMonTimes( rightMult[tracking[i]], ASCIIMon( j+1 ) );
      }
    }
  }
}
else // No sorting required
{
  // Convert the grid to monomial data
  for( i = 0; i < listLen; i++ ) // For each monomial
  {
    for( j = 0; j < (ULong) nOfGenerators; j++ ) // For each variable
    {
      if( grid[i][2*j] == 1 ) // LEFT Assigned
      {
        // Multiply on the left by a multiplicative variable
        leftMult[i] = fMonTimes( leftMult[i], ASCIIMon( j+1 ) );
      }
      if( grid[i][2*j+1] == 1 ) // RIGHT Assigned
      {
        // Multiply on the left by a multiplicative variable
        rightMult[i] = fMonTimes( rightMult[i], ASCIIMon( j+1 ) );
      }
    }
  }
}

// Convert the two arrays of FMons to an FMonPairList
for( i = 0; i < listLen; i++ )
  output = fMonPairListPush( leftMult[i], rightMult[i], output );

// Provide some final output information
if( pl > 3 )
{
  printf("OverlapDiv's Output (Left, Right) = \n");
```

```
    fMonPairListMultDisplay( fMonPairListRev( output ) );
  }

  // Return the reversed list (it was constructed in reverse)
  return fMonPairListFXRev( output );
}

/*
 * ====================================================
 * Polynomial Reduction and Basis Completion Functions
 * ====================================================
 */

/*
 * Function Name: IPolyReduce
 *
 * Overview: Reduces 1st arg w.r.t. 2nd arg (list) and 3rd arg (vars)
 *
 * Detail: Given a polynomial _poly_, this function involutively
 * reduces the polynomial with respect to the given FAlgList _list_
 * with associated left and right multiplicative variables _vars_.
 * The type of reduction (head reduction / full reduction) is
 * controlled by the global variable headReduce.
 * If IType > 3, we can take advantage of fast global reduction.
 *
 * External Variables Required: ULong nRed;
 * int IType, pl;
 * Global Variables Used: int headReduce;
 *
 */
FAlg
IPolyReduce( poly, list, vars )
FAlg poly;
FAlgList list;
FMonPairList vars;
{
  ULong i, numRules = fAlgListLength( list ), len,
        cutoffL, cutoffR, value, lenOrig, lenSub;
  FAlg LHSA[numRules], back = fAlgZero(), lead, upgrade;
  FMonPairList factors = fMonPairListNul;
  FMon LHSM[numRules], LHSVL[numRules], LHSVR[numRules],
       leadMonomial, leadLoopMonomial, JLeft, JRight,
       facLft, facRt, JMon;
  QInteger LHSQ[numRules], leadQ, leadLoopQ, lcmQ;
  short flag, toggle, M;
  int appears;

  // Catch special case list is empty
  if( !list ) return poly;

  // Convert the input list of polynomials to an array and
  // create arrays of lead monomials and lead coefficients
  for( i = 0; i < numRules; i++ )
```

```
{
  if( pl > 5 ) printf("Poly␣%u␣=␣%s\n", i+1, fAlgToStr( list -> first ) );
  LHSA[i] = list -> first;
  LHSM[i] = fAlgLeadMonom( list -> first );
  LHSQ[i] = fAlgLeadCoef( list -> first );
  if( IType < 3 ) // Using Local Division
  {
    // Create array of multiplicative variables
    LHSVL[i] = vars -> lft;
    LHSVR[i] = vars -> rt;
    vars = vars -> rest;
  }
  list = list -> rest;
}

// We will now recursively reduce every term in the polynomial
// until no more reductions are possible
while( fAlgIsZero( poly ) == (Bool) 0 )
{
  if( pl > 5 ) printf("Looking␣at␣Lead␣Term␣of␣%s\n", fAlgToStr( poly ) );
  toggle = 1; // Assume no reductions are possible to begin with
  lead = fAlgLeadTerm( poly );
  leadMonomial = fAlgLeadMonom( lead );
  leadQ = fAlgLeadCoef( lead );
  i = 0;

  while( i < numRules ) // For each polynomial in the list
  {
    if( IType >= 3 ) lenOrig = fMonLength( leadMonomial );
    leadLoopMonomial = LHSM[i]; // Pick a test monomial
    flag = 0;

    if( IType < 3 ) // Local Division
    {
      // Does the ith polynomial divide our polynomial?
      // If so, place all possible ways of doing this in factors
      factors = fMonDiv( leadMonomial, leadLoopMonomial, &flag );
    }
    else
    {
      if( IType == 5 )
        factors = fMonPairListNul; // No divisors w.r.t. Empty Division
      else
      {
        lenSub = fMonLength( leadLoopMonomial );

        // Check if a prefix/suffix is possible
        if( lenSub <= lenOrig )
        {
          if( IType == 3 ) // Left Division; look for Suffix
          {
            if( fMonEqual( leadLoopMonomial, fMonSuffix( leadMonomial, lenSub ) ) == (Bool) 1 )
            {
```

```
              if( lenOrig == lenSub )
                factors = fMonPairListSingle( fMonOne(), fMonOne() );
              else
                factors = fMonPairListSingle( fMonPrefix( leadMonomial, lenOrig-lenSub ), fMonOne() );
              flag = 1;
            }
          }
          else if( IType == 4 ) // Right Division; look for Prefix
          {
            if( fMonEqual( leadLoopMonomial, fMonPrefix( leadMonomial, lenSub ) ) == (Bool) 1 )
            {
              if( lenOrig == lenSub )
                factors = fMonPairListSingle( fMonOne(), fMonOne() );
              else
                factors = fMonPairListSingle( fMonOne(), fMonSuffix( leadMonomial, lenOrig-lenSub ) );
              flag = 1;
            }
          }
        }
      }
    }

    if( flag == 1 ) // i.e. leadLoopMonomial divides leadMonomial
    {
      M = 0; // Assume that the first conventional division is not an involutive division

      // While there are conventional divisions left to look at and
      // while none of these have yet proved to be involutive divisions
      while( ( fMonPairListLength( factors ) > 0 ) && ( M == 0 ) )
      {
        // Assume that this conventional division is an involutive division
        M = 1;
        if( IType < 3 ) // Local Division
        {
          // Extract the ith left & right multiplicative variables
          JLeft = LHSVL[i];
          JRight = LHSVR[i];

          // Extract the left and right factors
          facLft = factors -> lft;
          facRt = factors -> rt;

          // Test all variables in facLft for left multiplicability in the ith monomial
          len = fMonLength( facLft );

          // Decide whether one/all variables in facLft are left multiplicative
          if( MType == 1 ) // Right-most variable checked only
          {
            if( len > 0 )
            {
              JMon = fMonSuffix( facLft, 1 );
              appears = fMonIsMultiplicative( JMon, JLeft );
              // If the generator doesn't appear this is not an involutive division
```

```
        if( appears == 0 ) M = 0;
      }
    }
    else // All variables checked
    {
      while( len > 0 )
      {
        len = len − fMonLeadExp( facLft );
        // Extract a generator
        JMon = fMonPrefix( facLft, 1 );
        // Test to see if the generator appears in the list of left multiplicative variables
        appears = fMonIsMultiplicative( JMon, JLeft );
        // If the generator doesn't appear this is not an involutive division
        if( appears == 0 )
        {
          M = 0;
          break; // Exit from the while loop
        }
        facLft = fMonTailFac( facLft ); // Get ready to look at the next generator
      }
    }

    // Test all variables in facRt for right multiplicability in the ith monomial
    if( M == 1 )
    {
      len = fMonLength( facRt );

      // Decide whether one/all variables in facRt are left multiplicative
      if( MType == 1 ) // Left−most variable checked only
      {
        if( len > 0 )
        {
          JMon = fMonPrefix( facRt, 1 );
          appears = fMonIsMultiplicative( JMon, JRight );
          // If the generator doesn't appear this is not an involutive division
          if( appears == 0 ) M = 0;
        }
      }
      else // All variables checked
      {
        while( len > 0 )
        {
          len = len − fMonLeadExp( facRt );
          // Extract a generator
          JMon = fMonPrefix( facRt, 1 );
          // Test to see if the generator appears in the list of right multiplicative variables
          appears = fMonIsMultiplicative( JMon, JRight );
          // If the generator doesn't appear this is not an involutive division
          if( appears == 0 )
          {
            M = 0;
            break; // Exit from the while loop
          }
```

```
          facRt = fMonTailFac( facRt );
        }
      }
    }
  }
  else // Global division
  {
    M = 1; // Already potentially found an involutive divisor,
           // but include code below for reference

    /*
    // Obtain global cutoff positions
    if( IType == 3 ) LMultVars( leadLoopMonomial, &cutoffL, &cutoffR );
    else if( IType == 4 ) RMultVars( leadLoopMonomial, &cutoffL, &cutoffR );
    else EMultVars( leadLoopMonomial, &cutoffL, &cutoffR );
    if( pl > 4 ) printf("cutoff(%s) = (%u, %u)\n", fMonToStr( leadLoopMonomial ), cutoffL, cutoffR );

    // Extract the left and right factors
    facLft = factors -> lft;
    facRt = factors -> rt;

    // Test all variables in facLft for left multiplicability in the ith monomial
    len = fMonLength( facLft );

    // Decide whether one/all variables in facLft are left multiplicative
    if( MType == 1 ) // Right-most variable checked only
    {
      if( len > 0 )
      {
        JMon = fMonSuffix( facLft, 1 );
        value = ASCIIVal( fMonLeadVar( JMon ) );
        if( value > cutoffR ) M = 0;
      }
    }
    else // All variables checked
    {
      while( len > 0 )
      {
        len = len - fMonLeadExp( facLft );
        // Obtain the ASCII value of the next generator
        value = ASCIIVal( fMonLeadVar( facLft ) );
        if( value > cutoffR ) // If the generator is not left multiplicative
        {
          M = 0;
          break; // Exit from the while loop
        }
        facLft = fMonTailFac( facLft );
      }
    }

    // Test all variables in facRt for right multiplicability in the ith monomial
    len = fMonLength( facRt );

```

```
        // Decide whether one/all variables in facRt are left multiplicative
        if( MType == 1 ) // Left-most variable checked only
        {
          if( len > 0 )
          {
            value = ASCIIVal( fMonLeadVar( facRt ) );
            if( value < cutoffL ) M = 0;
          }
        }
        else // All variables checked
        {
          while( len > 0 )
          {
            len = len - fMonLeadExp( facRt );
            // Obtain the ASCII value of the next generator
            value = ASCIIVal( fMonLeadVar( facRt ) );
            if( value < cutoffL ) // If the generator is not right multiplicative
            {
              M = 0;
              break; // Exit from the while loop
            }
            facRt = fMonTailFac( facRt );
          }
        }
        */
      }

      // If this conventional division wasn't involutive, look at the next division
      if( M == 0 ) factors = factors -> rest;
    }

    // If an involutive division was found
    if( M == 1 )
    {
      if( pl > 1 ) nRed++; // Increase the number of reductions carried out
      if( pl > 5 ) printf("Found %s = (%s) * (%s) * (%s)\n", fMonToStr( leadMonomial ),
                          fMonToStr( factors -> lft ), fMonToStr( leadLoopMonomial ),
                          fMonToStr( factors -> rt ) );
      toggle = 0; // Indicate a reduction has been carried out to exit the loop
      leadLoopQ = LHSQ[i]; // Pick the divisor's leading coefficient
      lcmQ = AltLCMQInteger( leadQ, leadLoopQ ); // Pick 'nice' cancelling coefficients

      // Construct poly #i * -1 * coefficient to get lead terms the same
      upgrade = fAlgTimes( fAlgMonom( qOne(), factors -> lft ), LHSA[i] );
      upgrade = fAlgTimes( upgrade, fAlgMonom( qNegate( qDivide( lcmQ, leadLoopQ ) ), factors -> rt ) );

      // Add in poly * coefficient to cancel off the lead terms
      upgrade = fAlgPlus( upgrade, fAlgScaTimes( qDivide( lcmQ, leadQ ), poly ) );

      // We must also now multiply the current discarded remainder by a factor
      back = fAlgScaTimes( qDivide( lcmQ, leadQ ), back );
      poly = upgrade; // In the next iteration we will be reducing the new polynomial upgrade
      if( pl > 5 ) printf("New Word = %s; New Remainder = %s\n", fAlgToStr( poly ), fAlgToStr( back ) );
```

```
        }
      }
      if( toggle == 1 ) // The ith polynomial did not involutively divide poly
        i++;
      else // A reduction was carried out, exit the loop
        i = numRules;
    }

    if( toggle == 1 ) // No reductions were carried out; now look at the next term
    {
      // If only head reduction is required, return reducer
      if( headReduce == 1 ) return poly;

      // Otherwise add lead term to remainder and simplify the rest
      lead = fAlgLeadTerm( poly );
      back = fAlgPlus( back, lead );
      poly = fAlgPlus( fAlgNegate( lead ), poly );
      if( pl > 5 ) printf("New Remainder = %s\n", fAlgToStr( poly ) );
    }
  }

  return back; // Return the reduced and simplified polynomial
}

/*
 * Function Name: IAutoreduceFull
 *
 * Overview: Autoreduces an FAlgList recursively until no more reductions are possible
 *
 * Detail: This function involutively reduces each
 * member of an FAlgList w.r.t. all the other members
 * of the list, removing the polynomial from the list
 * if it is involutively reduced to 0. This process is
 * iterated until no more such reductions are possible.
 *
 * External Variables Required: int degRestrict, IType, pl, SType;
 * Global Variables Used: ULong d, twod;
 *
 */
FAlgList
IAutoreduceFull( input )
FAlgList input;
{
  FAlg oldPoly, newPoly;
  FAlgList new, old, oldCopy;
  FMonPairList vars = fMonPairListNul;
  ULong pos, pushPos, len = fAlgListLength( input );

  // If the input basis has more than one element
  if( len > 1 )
  {
    // Start by reducing the final element (working backwards means
    // that less work has to be done calculating multiplicative variables)
```

```
pos = len;
// If we are using a local division and the basis is sorted by DegRevLex,
// the last polynomial is irreducible so we do not have to consider it.
if( ( IType < 3 ) && ( SType == 1 ) ) pos --;

// Make a copy of the input basis for traversal
old = fAlgListCopy( input );

while( pos > 0 ) // For each polynomial in old
{
  // Extract the pos-th element of the basis
  oldPoly = fAlgListNumber( pos, old );
  if( pl > 2 ) printf("Looking at element p = %s of basis\n", fAlgToStr( oldPoly ) );

  // Construct basis without 'poly'
  oldCopy = fAlgListCopy( old ); // Make a copy of old

  // Calculate Multiplicative Variables if using a local division
  if( IType < 3 )
  {
    vars = OverlapDiv( oldCopy );
    vars = fMonPairListRemoveNumber( pos, vars );
  }

  new = fAlgListFXRem( old, oldPoly ); // Remove oldPoly from old
  old = fAlgListCopy( oldCopy ); // Restore old

  // To recap, _old_ is now unchanged whilst _new_ holds all
  // the elements of _old_ except _oldPoly_.

  // Involutively reduce the old polynomial w.r.t. the truncated list
  newPoly = IPolyReduce( oldPoly, new, vars );

  // If the polynomial did not reduce to 0
  if( fAlgIsZero( newPoly ) == (Bool) 0 )
  {
    // Divide the polynomial through by its GCD
    newPoly = findGCD( newPoly );
    if( pl > 2 ) printf("Reduced p to %s\n", fAlgToStr( newPoly ) );

    // Check for trivial ideal
    if( fAlgIsOne( newPoly ) == (Bool) 1 ) return fAlgListSingle( fAlgOne() );

    // If the old polynomial is equal to the new polynomial
    // (no reduction took place)
    if( fAlgEqual( oldPoly, newPoly ) == (Bool) 1 )
    {
      pos--; // We may proceed to look at the next polynomial
    }
    else // Otherwise some reduction took place so we have to start again
    {
      // If we are restricting prolongations based on degree,...
      if( degRestrict == 1 )
```

```
    {
      // ...and if the degree of the lead term of the new
      // polynomial exceeds the current bound...
      if( fMonLength( fAlgLeadMonom( newPoly ) ) > d )
      {
        // ...we must adjust the bound accordingly
        d = fMonLength( fAlgLeadMonom( newPoly ) );
        if( pl > 1 ) printf("New value of d = %u\n", d );
        twod = 2*d;
      }
    }

    // Add the new polynomial onto the list
    if( IType < 3 ) // Local division
    {
      if( SType == 1 ) // DegRevLex sorted
      {
        // Push the new polynomial onto the list
        old = fAlgListDegRevLexPushPosition( newPoly, new, &pushPos );
        // If it is inserted into the same position we may continue and look at the next polynomial
        if( pushPos == pos ) pos--;
        // If it is inserted into a later position we continue from one position above
        else if( pushPos > pos ) pos = pushPos - 1;
        // Note: the case pushPos < pos cannot occur
      }
      else if( SType == 2 ) // No sorting
      {
        // Push the new polynomial onto the end of the list
        old = fAlgListAppend( new, fAlgListSingle( newPoly ) );
        // Return to the end of the list minus one
        // (we know the last element is irreducible)
        pos = fAlgListLength( old ) - 1;
      }
      else // Sorted by main ordering
      {
        // Push the new polynomial onto the list
        old = fAlgListNormalPush( newPoly, new );
        // Return to the end of the list
        pos = fAlgListLength( old );
      }
    }
    else // Global division
    {
      // Push the new polynomial onto the end of the list
      old = fAlgListAppend( new, fAlgListSingle( newPoly ) );
      // Return to the end of the list minus one
      // (we know the last element is irreducible)
      pos = fAlgListLength( old ) - 1;
    }
  }
}
else // The polynomial reduced to zero
{
```

```
        // Remove the polynomial from the list
        old = fAlgListCopy( new );
        // Continue to look at the next element
        pos--;
        if( pl > 2 ) printf("Reduced p to 0\n");
      }
    }
  }
  else // The input basis is empty or consists of a single polynomial
    return input;

  // Return the fully autoreduced basis
  return old;
}

/*
 * Function Name: Seiler
 *
 * Overview: Implements Seiler's original algorithm for computing locally involutive bases
 *
 * Detail: Given a list of polynomials, this algorithm computes a
 * Locally Involutive Basis for the input basis by the following
 * iterative method: find all prolongations, choose the 'lowest'
 * one, autoreduce, find all prolongations, ...
 *
 * External Variables Required: int degRestrict, IType, nOfGenerators, pl, SType;
 * ULong nOfProlongations;
 * Global Variables Used: ULong d, twod;
 *
 */
FAlgList
Seiler( FBasis )
FAlgList FBasis;
{
  FAlgList H = fAlgListNul, HCopy = fAlgListNul, soFar = fAlgListNul, S;
  FAlg g, gNew, h;
  FMonPairList vars = fMonPairListNul, varsCopy,
               factors = fMonPairListNul;
  FMon all, LMh, Lmult, Rmult, nonMultiplicatives;
  ULong precount, count, degTest, len, i, cutoffL, cutoffR;
  short escape, degBound, flag, trip;

  if( pl > 0 ) printf("\nComputing an Involutive Basis...\n");

  if( IType < 3 ) // Local division
  {
    // Create a monomial containing all generators
    all = fMonOne();
    for( i = 1; i <= (ULong) nOfGenerators; i++ )
      all = fMonTimes( all, ASCIIMon( i ) );
  }

  // If prolongations are restricted by degree
```

```
if( degRestrict == 1 )
{
  d = maxDegree( FBasis ); // Initialise the value of d
  if( pl > 1 ) printf("Initial value of d = %u\n", d );

  /*
   * There is no point in looking at prolongations of length
   * 2*d or more as these cannot possibly be associated with
   * S-Polynomials - they are in effect 'disjoint overlaps'.
   */
  twod = 2*d;
}

// Turn head reduction off
headReduce = 0;

// Remove duplicates from the input basis
FBasis = fAlgListRemDups( FBasis );

// If the basis should be kept sorted, do the initial sorting now
if( ( IType < 3 ) && ( SType != 2 ) ) FBasis = fAlgListSort( FBasis, SType );

// Now Autoreduce FBasis and place the result in H
if( pl > 1 ) printf("Autoreducing...\n");
precount = fAlgListLength( FBasis ); // Determine size of basis before autoreduction
H = IAutoreduceFull( FBasis ); // Fully autoreduce the basis
count = fAlgListLength( H ); // Determine size of basis after autoreduction
if( ( pl > 0 ) && ( count < precount ) )
  printf("Autoreduction reduced the basis to size %u...\n", count );

// Check for trivial ideal
if( ( count == 1 ) & ( fAlgIsOne( H -> first ) == (Bool) 1 ) )
  return fAlgListSingle( fAlgOne() );

/*
 * soFar will store all polynomials that will appear in H
 * at any time so that we do not introduce duplicates into the set.
 * To begin with, all we have encountered are the polynomials
 * in the autoreduced input basis.
 */
soFar = fAlgListCopy( H );

escape = 1; // To enable the following while loop to begin
while( escape == 1 )
{
  if( IType < 3 ) // Calculate multiplicative variables for GBasis
  {
    vars = OverlapDiv( H );
    varsCopy = fMonPairListCopy( vars ); // Make a copy for traversal
  }

  HCopy = fAlgListCopy( H ); // Make a copy of H for traversal

```

```
// S will hold all the possible prolongations
S = fAlgListNul;

while( HCopy ) // For each $h \in H$
{
  h = HCopy -> first; // Extract a polynomial
  LMh = fAlgLeadMonom( h ); // Find the lead monomial
  if( pl == 3 ) printf("Analysing %s...\n", fMonToStr( LMh ) );
  if( pl > 3 ) printf("Analysing %s...\n", fAlgToStr( h ) );
  HCopy = HCopy -> rest; // Advance to the next polynomial

  // Assume to begin with that any prolongations of this polynomial are OK
  degBound = 0;
  if( degRestrict == 1 ) // If we are restricting prolongations by degree...
  {
    // ...and if the length of any prolongation of g exceeds the bound...
    if( fMonLength( LMh ) + 1 >= twod )
    {
      // ..ignore all prolongations involving this polynomial
      degBound = 1;
      if( pl > 2 ) printf("Degree of lead term exceeds 2*d-1\n");
      if( IType < 3 ) // Local division - advance to the next polynomial
        varsCopy = varsCopy -> rest;
    }
  }

  // Step 1 - find all prolongations

  if( ( IType < 3 ) && ( degBound == 0 ) ) // Local division
  {
    // Extract the left and right multiplicative variables for this polynomial
    Lmult = varsCopy -> lft;
    Rmult = varsCopy -> rt;
    varsCopy = varsCopy -> rest;

    // LEFT PROLONGATIONS

    // Construct the left nonmultiplicative variables
    nonMultiplicatives = all;
    while( fMonIsOne( Lmult ) != (Bool) 1 ) // For each left multiplicative variable
    {
      // Eliminate one multiplicative variable
      factors = fMonDivFirst( nonMultiplicatives, fMonPrefix( Lmult, 1 ), &flag );
      nonMultiplicatives = fMonTimes( factors -> lft, factors -> rt );
      Lmult = fMonRest( Lmult );
    }
    Lmult = nonMultiplicatives;
    // Find the number of left nonmultiplicative variables
    len = fMonLength( Lmult );

    // For each variable $x_i$ that is not Left Multiplicative for $LM(g)$
    for( i = 1; i <= len; i++ )
    {
```

```
        if( pl == 3 ) printf("Adding Left Prolongation by variable #%u to S...\n", i );
        if( pl > 3 ) printf("Adding Left Prolongation by %s to S...\n", fMonLeadVar( Lmult ) );
        S = fAlgListPush( fAlgTimes( fAlgMonom( qOne(), fMonPrefix( Lmult, 1 ) ), h ), S );
        Lmult = fMonRest( Lmult );
      }

      // RIGHT PROLONGATIONS

      // Construct the right nonmultiplicative variables
      nonMultiplicatives = all;
      while( fMonIsOne( Rmult ) != (Bool) 1 ) // For each right multiplicative variable
      {
        // Eliminate one multiplicative variable
        factors = fMonDivFirst( nonMultiplicatives, fMonPrefix( Rmult, 1 ), &flag );
        nonMultiplicatives = fMonTimes( factors -> lft, factors -> rt );
        Rmult = fMonRest( Rmult );
      }
      Rmult = nonMultiplicatives;
      // Find the number of right nonmultiplicative variables
      len = fMonLength( Rmult );

      // For each variable $x_i$ that is not Right Multiplicative for $LM(g)$
      for( i = 1; i <= len; i++ )
      {
        if( pl == 3 ) printf("Adding Right Prolongation by variable #%u to S...\n", i );
        if( pl > 3 ) printf("Adding Right Prolongation by %s to S...\n", fMonLeadVar( Rmult ) );
        S = fAlgListPush( fAlgTimes( h, fAlgMonom( qOne(), fMonPrefix( Rmult, 1 ) ) ), S );
        Rmult = fMonRest( Rmult );
      }
    }
    else if( ( IType >= 3 ) && ( degBound == 0 ) ) // Global division
    {
      // Find the multiplicative variables for this monomial
      if( IType == 3 ) LMultVars( LMh, &cutoffL, &cutoffR );
      else if( IType == 4 ) RMultVars( LMh, &cutoffL, &cutoffR );
      else EMultVars( LMh, &cutoffL, &cutoffR );
      if( pl > 4 ) printf("cutoff(%s) = (%u, %u)\n", fMonToStr( LMh ), cutoffL, cutoffR );

      // LEFT PROLONGATIONS

      // For each variable $x_i$ that is not Left Multiplicative for $LM(g)$
      for( i = cutoffR; i < (ULong) nOfGenerators; i++ )
      {
        // Construct a nonmultiplicative variable
        Lmult = ASCIIMon( i+1 );

        if( pl == 3 ) printf("Adding Left Prolongation by variable #%u to S...\n", i );
        if( pl > 3 ) printf("Adding Left Prolongation by %s to S...\n", fMonToStr( Lmult ) );
        S = fAlgListPush( fAlgTimes( fAlgMonom( qOne(), Lmult ), h ), S );
      }

      // RIGHT PROLONGATIONS

```

```
      // For each variable $x_i$ that is not Right Multiplicative for $LM(g)$
      for( i = 1; i < cutoffL; i++ )
      {
        // Construct a nonmultiplicative variable
        Rmult = ASCIIMon( i );

        if( pl == 3 ) printf("Adding Right Prolongation by variable #%u to S...\n", i-1 );
        if( pl > 3 ) printf("Adding Right Prolongation by %s to S...\n", fMonToStr( Rmult ) );
        S = fAlgListPush( fAlgTimes( h, fAlgMonom( qOne(), Rmult ) ), S );
      }
    }
  }

  // Step 2 - Find the lowest prolongation w.r.t. chosen monomial order

  // Turn head reduction on when finding a suitable prolongation
  headReduce = 1;

  // If there are no prolongations we may exit the loop
  if( !S ) escape = 0;
  else
  {
    // Sort the list of prolongations w.r.t. the chosen monomial order
    S = fAlgListSort( S, 3 );
    // Reverse the list so that the 'lowest' prolongation comes first
    S = fAlgListFXRev( S );

    // Obtain the first non-zero head-reduced element of the list
    g = S -> first; // Extract a prolongation
    trip = 0;
    // While there are prolongations left to look at and while we have
    // not yet found a non-zero head-reduced prolongation
    while( ( fAlgListLength( S ) > 0 ) && ( trip == 0 ) )
    {
      // Involutively head-reduce the prolongation
      gNew = IPolyReduce( g, H, vars );
      if( fAlgIsZero( gNew ) == (Bool) 0 ) // If the prolongation did not reduce to zero
      {
        // Turn off head reduction
        headReduce = 0;
        // 'Fully' involutively reduce
        gNew = IPolyReduce( gNew, H, vars );
        gNew = findGCD( gNew ); // Divide through by the GCD
        // Turn head reduction back on
        headReduce = 1;
        // If we have not encountered this polynomial before
        if( fAlgListIsMember( gNew, soFar ) == (Bool) 0 )
        {
          trip = 1; // We may exit the loop
          headReduce = 0; // We do not need head reduction any more
        }
        else // Otherwise we go on to look at the next prolongation
        {
```

```
          S = S -> rest; // Advance the list
          if( S ) g = S -> first; // If there are any more prolongations extract one
        }
      }
      else // Otherwise we go on to look at the next prolongation
      {
        S = S -> rest; // Advance the list
        if( S ) g = S -> first; // If there are any more prolongations extract one
      }
    }

    // If no suitable prolongations were found we may exit the loop
    if( !S ) escape = 0;
    else
    {
      // Step 3 - Add the polynomial to the basis

      if( pl > 2 ) printf("First Non-Zero Reduced Prolongation = %s\n", fAlgToStr( g ) );
      if( pl > 2 ) printf("Prolongation after reduction = %s\n", fAlgToStr( gNew ) );
      nOfProlongations++; // Increase the counter for the number of prolongations processed

      // Check for trivial ideal
      if( fAlgIsOne( gNew ) == (Bool) 1 ) return fAlgListSingle( fAlgOne() );

      // Adjust the prolongation degree bound if necessary
      if( degRestrict == 1 )
      {
        if( fAlgEqual( g, gNew ) == (Bool) 0 ) // If the polynomial was reduced...
        {
          degTest = fMonLength( fAlgLeadMonom( gNew ) );
          if( degTest > d ) // ...and if the degree of the new polynomial exceeds the bound...
          {
            // ...adjust the bound accordingly
            d = degTest;
            if( pl > 1 ) printf("New value of d = %u\n", d );
            twod = 2*d;
          }
        }
      }

      // Push the new polynomial onto the list
      if( IType < 3 ) // Local division
      {
        if( SType == 1 ) H = fAlgListDegRevLexPush( gNew, H ); // DegRevLex sort
        else if( SType == 2 ) H = fAlgListAppend( H, fAlgListSingle( gNew ) ); // No sorting - just append
        else H = fAlgListNormalPush( gNew, H ); // Sort by monomial ordering
      }
      else H = fAlgListAppend( H, fAlgListSingle( gNew ) ); // Just append onto end

      count++; // Increase the counter for the number of polynomials in the basis
      if( pl > 1 ) printf("Added Polynomial #%u to Basis, namely\n %s \n", count, fAlgToStr( gNew ) );
      if( pl == 1 ) printf("Added Polynomial #%u to Basis...\n", count );
      // Indicate that we have encountered a new polynomial for future reference
```

```
        soFar = fAlgListPush( gNew, soFar );

        // Step 4 − Autoreduce

        precount = count; // Determine size of basis before autoreduction
        H = IAutoreduceFull( H ); // Fully autoreduce the basis
        count = fAlgListLength( H ); // Determine size of basis after autoreduction
        if( ( pl > 0 ) && ( count < precount ) )
          printf("Autoreduction␣reduced␣the␣basis␣to␣size␣%u...\n", count );

        // Check for trivial ideal
        if( ( count == 1 ) && ( fAlgIsOne( H −> first ) == (Bool) 1 ) )
          return fAlgListSingle( fAlgOne() );
      }
    }
  }
  if( pl > 0 ) printf("...Involutive␣Basis␣Computed.\n");

  headReduce = 0; // Reset the value of headReduce
  return H;
}

/*
 * Function Name: Gerdt
 *
 * Overview: Implements Gerdt's advanced algorithm for computing locally involutive bases
 *
 * Detail: Given a list of polynomials, this algorithm computes a
 * Locally Involutive Basis for the input basis using the method
 * outlined in the paper "Involutive Division Technique:
 * Some generalisations and optimisations" by V. P. Gerdt.
 *
 * External Variables Required: int degRestrict, IType, nOfGenerators, pl, SType;
 * ULong nOfProlongations;
 * Global Variables Used: ULong d, twod;
 * int headReduce;
 *
 */
FAlgList
Gerdt( FBasis )
FAlgList FBasis;
{
  FAlgList GBasis = fAlgListNul, soFar = fAlgListNul,
           Tp = fAlgListNul, Qp = fAlgListNul,
           Tp2 = fAlgListNul, Qp2 = fAlgListNul;
  FAlg f, g, h, gDotx, candidatePoly, testPoly;
  FMonPairList Tv = fMonPairListNul, Qv = fMonPairListNul,
               Tv2 = fMonPairListNul, vars = fMonPairListNul;
  FMonList Tm = fMonListNul, Qm = fMonListNul,
           Tm2 = fMonListNul;
  FMonPair P, fVars, gVars, hVars;
  FMon PL, PR, fVarsL, fVarsR, gVarsL, gVarsR, hVarsL, hVarsR,
       LMf, LMg, LMh, all, DL, DR, gen, NML, NMR, u,
```

```
      candidateVariable, mult, compare;
ULong i, j, candidatePos, count, cutoffL, cutoffR,
       degTest, lowest, precount, pos;
short add, escape, LorR;
Bool balance;

if( pl > 0 ) printf("\nComputing an Involutive Basis...\n");

if( IType < 3 ) // Local division
{
  // Create a monomial containing all generators
  all = fMonOne();
  for( i = 1; i <= (ULong) nOfGenerators; i++ )
    all = fMonTimes( all, ASCIIMon( i ) );
}

// If prolongations are restricted by degree
if( degRestrict == 1 )
{
  d = maxDegree( FBasis ); // Initialise the value of d
  if( pl > 1 ) printf("Initial value of d = %u\n", d );

  /*
   * There is no point in looking at prolongations of length
   * 2*d or more as these cannot possibly be associated with
   * S-Polynomials - they are in effect 'disjoint overlaps'.
   */
  twod = 2*d;
}

// Turn head reduction off
headReduce = 0;

// Remove duplicates from the input basis
FBasis = fAlgListRemDups( FBasis );

// If the basis should be kept sorted, do the initial sorting now
if( ( IType < 3 ) && ( SType != 2 ) ) FBasis = fAlgListSort( FBasis, SType );

// Now Autoreduce FBasis and place the result in FBasis
if( pl > 1 ) printf("Autoreducing...\n");
precount = fAlgListLength( FBasis ); // Determine size of basis before autoreduction
FBasis = IAutoreduceFull( FBasis ); // Fully autoreduce the basis
count = fAlgListLength( FBasis ); // Determine size of basis after autoreduction
if( ( pl > 0 ) && ( count < precount ) )
  printf("Autoreduction reduced the basis to size %u...\n", count );

// Check for trivial ideal
if( ( count == 1 ) & ( fAlgIsOne( FBasis -> first ) == (Bool) 1 ) )
  return fAlgListSingle( fAlgOne() );

/*
 * soFar will store all polynomials that will appear
```

```
 * at any time so that we do not introduce duplicates into the set.
 * To begin with, all we have encountered are the polynomials
 * in the autoreduced input basis.
 */
soFar = fAlgListCopy( FBasis );

// Choose g \in F with lowest LM(g) w.r.t. <
g = fAlgListNumber( ( fAlgListLowest( FBasis ) ), FBasis );

// Add entry (g, LM(g), (\emptyset, \emptyset)) to T
Tp = fAlgListPush( g, Tp );
Tm = fMonListPush( fAlgLeadMonom( g ), Tm );
Tv = fMonPairListPush( fMonOne(), fMonOne(), Tv );

// Add entry to G
GBasis = fAlgListPush( g, GBasis );
if( pl > 1 ) printf("Adding %s to G (%u)...\n", fAlgToStr( g ), fAlgListLength( GBasis ) );
else if( pl == 1 ) printf("Added a first polynomial to G...\n");

// For each f \in FBasis\setminus {g}...
while( FBasis )
{
  f = FBasis -> first;
  if( fAlgEqual( g, f ) == (Bool) 0 )
  {
    // Add entry (f, LM(f), (\emptyset, \emptyset)) to Q
    Qp = fAlgListPush( f, Qp );
    Qm = fMonListPush( fAlgLeadMonom( f ), Qm );
    Qv = fMonPairListPush( fMonOne(), fMonOne(), Qv );
  }
  FBasis = FBasis -> rest;
}
if( pl > 3 ) printf("Constructed Q...\n");

do // Repeat until Q is empty
{
  h = fAlgZero();

  // While Q is not empty and h is not equal to 0
  while( ( fAlgListLength( Qp ) > 0 ) && ( fAlgIsZero( h ) == (Bool) 1 ) )
  {
    // Choose the g in (g, u (PL, PR) ) \in Q with lowest LM(g) w.r.t. <
    lowest = fAlgListLowest( Qp );
    g = fAlgListNumber( lowest, Qp );
    u = fMonListNumber( lowest, Qm );
    P = fMonPairListNumber( lowest, Qv );
    if( pl > 2 ) printf("Testing g = %s...\n", fAlgToStr( g ) );

    // Remove entry from Q
    Qp = fAlgListRemoveNumber( lowest, Qp );
    Qm = fMonListRemoveNumber( lowest, Qm );
    Qv = fMonPairListRemoveNumber( lowest, Qv );

```

```
    if( IType < 3 ) // Find Local Multiplicative Variables for GBasis
      vars = OverlapDiv( GBasis );

    // If the criterion is false... (to be implemented in the future...)
    // if( NCcriterion( g, u, Tp, Tm, GBasis, vars ) == 0 )
    {
      // ...then find the normal form of g w.r.t. GBasis
      soFar = fAlgListPush( g, soFar );
      h = IPolyReduce( g, GBasis, vars ); // Find the involutive normal form
      h = findGCD( h ); // Divide through by the GCD
      if( pl > 2 ) printf("...Reduced g to h = %s...\n", fAlgToStr( h ) );
    }
    // else if( pl > 2 ) printf("... Criterion used to discard g...\n");
  }

  // If h \neq 0
  if( fAlgIsZero( h ) == (Bool) 0 )
  {
    // Add h to GBasis and recalculate multiplicative variables if necessary
    if( IType < 3 )
    {
      pos = 1;
      if( SType == 1 ) GBasis = fAlgListDegRevLexPushPosition( h, GBasis, &pos ); // DegRevLex sort
      else if( SType == 2 ) GBasis = fAlgListAppend( GBasis, fAlgListSingle( h ) ); // No sorting - just append
      else GBasis = fAlgListNormalPush( h, GBasis ); // Sort by monomial ordering

      vars = OverlapDiv( GBasis ); // Full recalculate
    }
    else GBasis = fAlgListAppend( GBasis, fAlgListSingle( h ) ); // Just append onto end

    if( pl > 1 ) printf("Added %s to G (%u)...\n", fAlgToStr( h ), fAlgListLength( GBasis ) );
    else if( pl == 1 ) printf("Added a polynomial to G       (%u)...\n", fAlgListLength( GBasis ));

    LMh = fAlgLeadMonom( h );

    if( degRestrict == 1 ) // If we are restricting prolongations by degree...
    {
      degTest = fMonLength( LMh );
      if( degTest > d ) // ...and if the degree of the new polynomial exceeds the bound...
      {
        // ...adjust the bound accordingly
        d = degTest;
        if( pl > 1 ) printf("New value of d = %u\n", d );
        twod = 2*d;
      }
    }

    // If LM(h) == LM(g)
    if( fMonEqual( fAlgLeadMonom( g ), LMh ) == (Bool) 1 )
    {
      // Add entry to T
      Tp = fAlgListPush( h, Tp );
      Tm = fMonListPush( u, Tm );
```

```
    if( pl > 4 ) printf("Modifying T (size %u)...\n", fAlgListLength( Tp ) );

    // Find intersection of P and NM_I(h, G)
    // (Note: NM_I(h, G) = nonmultiplicative variables)
    PL = P.lft;
    PR = P.rt;

    if( IType < 3 ) // Local division
    {
      // Find NM_I(h, GBasis)
      pos = fAlgListPosition( h, GBasis );
      hVars = fMonPairListNumber( pos, vars );
      hVarsL = hVars.lft;
      hVarsR = hVars.rt;

      NML = fMonOne();
      NMR = fMonOne();
      j = 1;

      // Calculate the intersection
      while( j <= (ULong) nOfGenerators )
      {
        gen = ASCIIMon( j );

        // If gen appears in PL (nonmultiplicatives) but not in hVarsL (multiplicatives)
        if ( ( fMonIsMultiplicative( gen, PL ) == 1 ) && ( fMonIsMultiplicative( gen, hVarsL ) == 0 ) )
          NML = fMonTimes( NML, gen ); // gen appears in the left intersection
        // If gen appears in PR (nonmultiplicatives) but not in hVarsR (multiplicatives)
        if ( ( fMonIsMultiplicative( gen, PR ) == 1 ) && ( fMonIsMultiplicative( gen, hVarsR ) == 0 ) )
          NMR = fMonTimes( NMR, gen ); // gen appears in the right intersection

        j++; // Get ready to look at the next variable
      }
    }
    else if( IType >= 3 ) // Global division
    {
      // Find the multiplicative variables
      if( IType == 3 ) LMultVars( LMh, &cutoffL, &cutoffR );
      else if( IType == 4 ) RMultVars( LMh, &cutoffL, &cutoffR );
      else EMultVars( LMh, &cutoffL, &cutoffR );
      NML = fMonOne();
      NMR = fMonOne();

      // Calculate the left intersection
      for( j = cutoffR+1; j <= (ULong) nOfGenerators; j++ )
      {
        gen = ASCIIMon( j ); // Obtain a nonmultiplicative variable
        // If it appears in PL it appears in the intersection
        if( fMonIsMultiplicative( gen, PL ) == 1 )
          NML = fMonTimes( NML, gen );
      }

      // Calculate the right intersection
```

```
        for( j = 1; j < cutoffL; j++ )
        {
          gen = ASCIIMon( j ); // Obtain a nonmultiplicative variable
          // If it appears in PR it appears in the intersection
          if( fMonIsMultiplicative( gen, PR ) == 1 )
            NMR = fMonTimes( NMR, gen );
        }
      }

      // Add an entry to Tv
      Tv = fMonPairListPush( NML, NMR, Tv );
    }
    else // Add entry to T and adjust the lists
    {
      // Add entry to T
      Tp = fAlgListPush( h, Tp );
      Tm = fMonListPush( LMh, Tm );
      Tv = fMonPairListPush( fMonOne(), fMonOne(), Tv );
      if( pl > 4 ) printf("Modifying T (size %u)...\n", fAlgListLength( Tp ) );

      // Set up lists for next operation
      Tp2 = fAlgListNul;
      Tm2 = fMonListNul;
      Tv2 = fMonPairListNul;

      // For each (f, v, (DL, DR)) \in T
      if( pl > 4 ) printf("Adjusting Multiplicative Variables...\n");
      while( Tp )
      {
        f = Tp -> first; // Extract a polynomial
        LMf = fAlgLeadMonom( f );

        if( pl > 4 ) printf("Testing (%s, %s)\n", fMonToStr( LMh ), fMonToStr( LMf ) );

        // If LM(h) < LM(f)
        if( theOrdFun( LMh, LMf ) == (Bool) 1 )
        {
          // Add entry to Q
          Qp = fAlgListPush( Tp -> first, Qp );
          Qm = fMonListPush( Tm -> first, Qm );
          Qv = fMonPairListPush( Tv -> lft, Tv -> rt, Qv );

          // Discard f from GBasis
          GBasis = fAlgListFXRem( GBasis, f );
          if( pl > 1 ) printf("Discarded %s from G (%u)...\n", fAlgToStr( f ), fAlgListLength( GBasis ) );
          else if ( pl == 1 ) printf("Discarded a polynomial from G (%u)...\n", fAlgListLength( GBasis ) );
        }
        else
        {
          // Keep entry in T
          Tp2 = fAlgListPush( Tp -> first, Tp2 );
          Tm2 = fMonListPush( Tm -> first, Tm2 );
          Tv2 = fMonPairListPush( Tv -> lft, Tv -> rt, Tv2 );
```

```
      }
      // Advance the lists to the next entry
      Tp = Tp -> rest;
      Tm = Tm -> rest;
      Tv = Tv -> rest;
    }

    // Set up lists for next operation
    Tp = fAlgListNul;
    Tm = fMonListNul;
    Tv = fMonPairListNul;

    // Recalculate multiplicative variables
    if( IType < 3 ) vars = OverlapDiv( GBasis );

    // For each (f, v, (DL, DR)) \in T
    while( Tp2 )
    {
      // Keep f and v as they are
      f = Tp2 -> first;
      Tp = fAlgListPush( f, Tp );
      Tm = fMonListPush( Tm2 -> first, Tm );
      DL = Tv2 -> lft;
      DR = Tv2 -> rt;

      // Find intersection of D and NM_I(f, G)
      if( IType < 3 ) // Local division
      {
        // Find NM_I(f, GBasis)
        pos = fAlgListPosition( f, GBasis );
        fVars = fMonPairListNumber( pos, vars );
        fVarsL = fVars.lft;
        fVarsR = fVars.rt;

        NML = fMonOne();
        NMR = fMonOne();
        j = 1;

        // Calculate the intersection
        while( j <= (ULong) nOfGenerators )
        {
          gen = ASCIIMon( j );

          // If gen appears in DL (nonmultiplicatives) but not in fVarsL (multiplicatives)
          if ( ( fMonIsMultiplicative( gen, DL ) == 1 ) && ( fMonIsMultiplicative( gen, fVarsL ) == 0 ) )
            NML = fMonTimes( NML, gen ); // gen appears in the left intersection
          // If gen appears in DR (nonmultiplicatives) but not in fVarsR (multiplicatives)
          if ( ( fMonIsMultiplicative( gen, DR ) == 1 ) && ( fMonIsMultiplicative( gen, fVarsR ) == 0 ) )
            NMR = fMonTimes( NMR, gen ); // gen appears in the right intersection

          j++; // Get ready to look at the next variable
        }
      }
```

```
          else if ( IType >= 3 ) // Global division
          {
            // Find the multiplicative variables
            if( IType == 3 ) LMultVars( fAlgLeadMonom( f ), &cutoffL, &cutoffR );
            else if( IType == 4 ) RMultVars( fAlgLeadMonom( f ), &cutoffL, &cutoffR );
            else EMultVars( fAlgLeadMonom( f ), &cutoffL, &cutoffR );
            NML = fMonOne();
            NMR = fMonOne();

            // Calculate the left intersection
            for( j = cutoffR+1; j <= (ULong) nOfGenerators; j++ )
            {
              gen = ASCIIMon( j ); // Obtain a nonmultiplicative variable
              // If it appears in DL it appears in the intersection
              if( fMonIsMultiplicative( gen, DL ) == 1 )
                NML = fMonTimes( NML, gen );
            }

            // Calculate the right intersection
            for( j = 1; j < cutoffL; j++ )
            {
              gen = ASCIIMon( j ); // Obtain a nonmultiplicative variable
              // If it appears in DR it appears in the intersection
              if( fMonIsMultiplicative( gen, DR ) == 1 )
                NMR = fMonTimes( NMR, gen );
            }
          }

          // Add the nonmultiplicative variables to Tv
          Tv = fMonPairListPush( NML, NMR, Tv );

          // Advance the lists
          Tp2 = Tp2 -> rest;
          Tm2 = Tm2 -> rest;
          Tv2 = Tv2 -> rest;
        }
      }
    }

    // Recalculate multiplicative variables
    if( IType < 3 ) vars = OverlapDiv( GBasis );

    // While exist (g, u, (PL, PR)) \in T and x \in NM_I(g, GBasis)\P and,
    // if Q \neq \emptyset, s.t. LM(prolongation) < LM(f) for all f in
    // (f, v, (DL, DR)) \in Q do...
    escape = 0;
    while( escape == 0 )
    {
      // Construct a candidate set for (g, u, (PL, PR)), x
      if( pl > 3 ) printf("Finding candidates for (g, u, (PL, PR)), x...\n");

      // Initialise variables
```

```
    Tp2 = fAlgListCopy( Tp );
    Tm2 = fMonListCopy( Tm );
    Tv2 = fMonPairListCopy( Tv );
    candidatePos = 0;
    candidatePoly = fAlgZero();
    candidateVariable = fMonOne();
    LorR = 0;
    if( IType < 3 ) vars = OverlapDiv( GBasis );

    // For each (g, u, (PL, PR)) in T
    i = 1;
    while( Tp2 )
    {
      // Extract information about the first entry in T
      g = Tp2 -> first;
      LMg = fAlgLeadMonom( g );
      PL = Tv2 -> lft;
      PR = Tv2 -> rt;

      // Advance the copy of T
      Tp2 = Tp2 -> rest;
      Tm2 = Tm2 -> rest;
      Tv2 = Tv2 -> rest;

      if( IType < 3 ) // Local division
      {
        pos = fAlgListPosition( g, GBasis );
        gVars = fMonPairListNumber( pos, vars );
        gVarsL = gVars.lft;
        gVarsR = gVars.rt;

        j = 1;
        while( j <= (ULong) nOfGenerators ) // For each generator
        {
          gen = ASCIIMon( j );

          // LEFT PROLONGATIONS

          // Look for nonmultiplicative variables not in PL (unprocessed)
          if( ( fMonIsMultiplicative( gen, PL ) == 0 ) && ( fMonIsMultiplicative( gen, gVarsL ) == 0 ) )
          {
            add = 1; // Candidate found
            mult = fMonTimes( gen, fAlgLeadMonom( g ) ); // Construct x.g

            // If Q is not empty
            if( Qp )
            {
              // Make sure that LM(x.g) < LM(f) for all f in (f, v, D) \in Q
              Qp2 = fAlgListCopy( Qp ); // Make a copy of Q for processing
              while( ( fAlgListLength( Qp2 ) > 0 ) && ( add == 1 ) ) // For all f in (f, v, D) \in Q
              {
                // Extract a lead monomial
                compare = fAlgLeadMonom( Qp2 -> first );
```

```
      Qp2 = Qp2 -> rest;

      // If LM(x.g) not less than LM(f) ignore this candidate
      if( theOrdFun( mult, compare ) == (Bool) 0 ) add = 0;
    }
  }

  if( add == 1 ) // Candidate found for (g, u, (PL, PR)), x
  {
    if( candidatePos > 0 ) // This is not the first candidate tried
      // Returns 1 if mult < fAlgLeadMonom( candidatePoly )
      balance = theOrdFun( mult, fAlgLeadMonom( candidatePoly ) );

    // If we are restricting prolongations by degree
    if( degRestrict == 1 )
    {
      // If the degree bound is not exceeded and the candidate is valid
      if( ( fMonLength( LMg ) + 1 < twod ) && ( ( balance == (Bool) 1 ) || ( candidatePos == 0 ) ) )
      {
        // Construct a candidate prolongation
        testPoly = fAlgTimes( fAlgMonom( qOne(), gen ), g );
        // If we have not yet encountered this polynomial
        if( fAlgListIsMember( testPoly, soFar ) == (Bool) 0 )
        {
          // We have found a new candidate
          candidatePos = i;
          candidatePoly = testPoly;
          candidateVariable = gen;
          LorR = 0; // Left prolongation
        }
      }
    }
    // If we are not restricting prolongations by degree, proceed if
    // the candidate is valid (if this is the first candidate
    // encountered or LM(x.g) < LM(current candidate))
    else if( ( balance == (Bool) 1 ) | ( candidatePos == 0 ) )
    {
      // Construct a candidate prolongation
      testPoly = fAlgTimes( fAlgMonom( qOne(), gen ), g );
      // If we have not yet encountered this polynomial
      if( fAlgListIsMember( testPoly, soFar ) == (Bool) 0 )
      {
        // We have found a new candidate
        candidatePos = i;
        candidatePoly = testPoly;
        candidateVariable = gen;
        LorR = 0; // Left prolongation
      }
    }
  }
}

// RIGHT PROLONGATIONS
```

```
        // Look for nonmultiplicative variables not in PR (unprocessed)
        if( ( fMonIsMultiplicative( gen, PR ) == 0 ) && ( fMonIsMultiplicative( gen, gVarsR ) == 0 ) )
        {
          add = 1; // Candidate found
          mult = fMonTimes( fAlgLeadMonom( g ), gen ); // Construct g.x

          // If Q is not empty
          if( Qp )
          {
            // Make sure that LM(g.x) < LM(f) for all f in (f, v, D) \in Q
            Qp2 = fAlgListCopy( Qp ); // Make a copy of Q for processing

            while( ( fAlgListLength( Qp2 ) > 0 ) && ( add == 1 ) ) // For all f in (f, v, D) \in Q
            {
              // Extract a lead monomial
              compare = fAlgLeadMonom( Qp2 -> first );
              Qp2 = Qp2 -> rest;

              // If LM(g.x) not less than LM(f) ignore this candidate
              if( theOrdFun( mult, compare ) == (Bool) 0 ) add = 0;
            }
          }

          if( add == 1 ) // Candidate found for (g, u, (PL, PR)), x
          {
            if( candidatePos > 0 ) // This is not the first candidate tried
              // Returns 1 if mult < fAlgLeadMonom( candidatePoly )
              balance = theOrdFun( mult, fAlgLeadMonom( candidatePoly ) );

            // If we are restricting prolongations by degree
            if( degRestrict == 1 )
            {
              // If the degree bound is not exceeded and the candidate is valid
              if( ( fMonLength( LMg ) + 1 < twod ) && ( ( balance == (Bool) 1 ) || ( candidatePos == 0 ) ) )
              {
                // Construct a candidate prolongation
                testPoly = fAlgTimes( g, fAlgMonom( qOne(), gen ) );

                // If we have not yet encountered this polynomial
                if( fAlgListIsMember( testPoly, soFar ) == (Bool) 0 )
                {
                  // We have found a new candidate
                  candidatePos = i;
                  candidatePoly = testPoly;
                  candidateVariable = gen;
                  LorR = 1; // Right prolongation
                }
              }
            }
            // If we are not restricting prolongations by degree, proceed if
            // the candidate is valid (if this is the first candidate
            // encountered or LM(g.x) < LM(current candidate))
```

```
        else if( ( balance == (Bool) 1 ) | ( candidatePos == 0 ) )
        {
          // Construct a candidate prolongation
          testPoly = fAlgTimes( g, fAlgMonom( qOne(), gen ) );
          // If we have not yet encountered this polynomial
          if( fAlgListIsMember( testPoly, soFar ) == (Bool) 0 )
          {
            // We have found a new candidate
            candidatePos = i;
            candidatePoly = testPoly;
            candidateVariable = gen;
            LorR = 1; // Right prolongation
          }
        }
      }
    }
    j++; // Move onto the next variable
  }
}
else if( IType >= 3 ) // Global division
{
  // Obtain the multiplicative variables for this polynomial
  if( IType == 3 ) LMultVars( fAlgLeadMonom( g ), &cutoffL, &cutoffR );
  else if( IType == 4 ) RMultVars( fAlgLeadMonom( g ), &cutoffL, &cutoffR );
  else EMultVars( fAlgLeadMonom( g ), &cutoffL, &cutoffR );

  // LEFT PROLONGATIONS

  // For each left nonmultiplicative variable
  for( j = cutoffR+1; j <= (ULong) nOfGenerators; j++ )
  {
    gen = ASCIIMon( j );

    if( fMonIsMultiplicative( gen, PL ) == 0 ) // Not in P (unprocessed)
    {
      add = 1; // Candidate found
      mult = fMonTimes( gen, fAlgLeadMonom( g ) ); // Construct x.g

      // If Q is not empty
      if( Qp )
      {
        // Make sure that LM(x.g) < LM(f) for all f in (f, v, D) \in Q
        Qp2 = fAlgListCopy( Qp ); // Make a copy of Q for processing

        while( ( fAlgListLength( Qp2 ) > 0 ) && ( add == 1 ) ) // For all f in (f, v, D) \in Q
        {
          // Extract a lead monomial
          compare = fAlgLeadMonom( Qp2 -> first );
          Qp2 = Qp2 -> rest;

          // If LM(x.g) not less than LM(f) ignore this candidate
          if( theOrdFun( mult, compare ) == (Bool) 0 ) add = 0;
        }
```

```
        }

        if( add == 1 ) // Candidate found for (g, u, (PL, PR)), x
        {
          if( candidatePos > 0 ) // This is not the first candidate tried
            // Returns 1 if mult < fAlgLeadMonom( candidatePoly )
            balance = theOrdFun( mult, fAlgLeadMonom( candidatePoly ) );

          // If we are restricting prolongations by degree
          if( degRestrict == 1 )
          {
            // If the degree bound is not exceeded and the candidate is valid
            if( ( fMonLength( LMg ) + 1 < twod ) && ( ( balance == (Bool) 1 ) || ( candidatePos == 0 ) ) )
            {
              // Construct a candidate prolongation
              testPoly = fAlgTimes( fAlgMonom( qOne(), gen ), g );
              // If we have not yet encountered this polynomial
              if( fAlgListIsMember( testPoly, soFar ) == (Bool) 0 )
              {
                // We have found a new candidate
                candidatePos = i;
                candidatePoly = testPoly;
                candidateVariable = gen;
                LorR = 0; // Left prolongation
              }
            }
          }
          // If we are not restricting prolongations by degree, proceed if
          // the candidate is valid (if this is the first candidate
          // encountered or LM(x.g) < LM(current candidate))
          else if( ( balance == (Bool) 1 ) | ( candidatePos == 0 ) )
          {
            // Construct a candidate prolongation
            testPoly = fAlgTimes( fAlgMonom( qOne(), gen ), g );
            // If we have not yet encountered this polynomial
            if( fAlgListIsMember( testPoly, soFar ) == (Bool) 0 )
            {
              // We have found a new candidate
              candidatePos = i;
              candidatePoly = testPoly;
              candidateVariable = gen;
              LorR = 0; // Left prolongation
            }
          }
        }
      }
    }

    // RIGHT PROLONGATIONS

    // For each right nonmultiplicative variable
    for( j = 1; j < cutoffL; j++ )
    {
```

```
          gen = ASCIIMon( j );
          mult = fMonTimes( fAlgLeadMonom( g ), gen ); // Construct g.x

          if( fMonIsMultiplicative( gen, PR ) == 0 ) // Not in P (unprocessed)
          {
            add = 1; // Candidate found

            // If Q is not empty
            if( Qp )
            {
              // Make sure that LM(g.x) < LM(f) for all f in (f, v, D) \in Q
              Qp2 = fAlgListCopy( Qp ); // Make a copy of Q for processing

              while( ( fAlgListLength( Qp2 ) > 0 ) && ( add == 1 ) ) // For all f in (f, v, D) \in Q
              {
                // Extract a lead monomial
                compare = fAlgLeadMonom( Qp2 -> first );
                Qp2 = Qp2 -> rest;

                // If LM(g.x) not less than LM(f) ignore this candidate
                if( theOrdFun( mult, compare ) == (Bool) 0 ) add = 0;
              }
            }

            if( add == 1 ) // Candidate found for (g, u, (PL, PR)), x
            {
              if( candidatePos > 0 ) // This is not the first candidate tried
                // Returns 1 if mult < fAlgLeadMonom( candidatePoly )
                balance = theOrdFun( mult, fAlgLeadMonom( candidatePoly ) );

              // If we are restricting prolongations by degree
              if( degRestrict == 1 )
              {
                // If the degree bound is not exceeded and the candidate is valid
                if( ( fMonLength( LMg ) + 1 < twod ) && ( ( balance == (Bool) 1 ) || ( candidatePos == 0 ) ) )
                {
                  // Construct a candidate prolongation
                  testPoly = fAlgTimes( g, fAlgMonom( qOne(), gen ) );
                  // If we have not yet encountered this polynomial
                  if( fAlgListIsMember( testPoly, soFar ) == (Bool) 0 )
                  {
                    // We have found a new candidate
                    candidatePos = i;
                    candidatePoly = testPoly;
                    candidateVariable = gen;
                    LorR = 1; // Right prolongation
                  }
                }
              }
              // If we are not restricting prolongations by degree, proceed if
              // the candidate is valid (if this is the first candidate
              // encountered or LM(g.x) < LM(current candidate))
              else if( ( balance == (Bool) 1 ) | ( candidatePos == 0 ) )
```

```
          {
            // Construct a candidate prolongation
            testPoly = fAlgTimes( g, fAlgMonom( qOne(), gen ) );
            // If we have not yet encountered this polynomial
            if( fAlgListIsMember( testPoly, soFar ) == (Bool) 0 )
            {
              // We have found a new candidate
              candidatePos = i;
              candidatePoly = testPoly;
              candidateVariable = gen;
              LorR = 1; // Right prolongation
            }
          }
        }
      }
    }
  }
  i++; // Move onto the next polynomial
}
if( pl > 3 ) printf("...Element %u chosen as the candidate (0 = none found).\n", candidatePos );

// If there is a candidate
if( candidatePos > 0 )
{
  // Construct the candidate
  g = fAlgListNumber( candidatePos, Tp );
  u = fMonListNumber( candidatePos, Tm );
  P = fMonPairListNumber( candidatePos, Tv );
  if( pl > 2 )
  {
    if( LorR == 0 )
      printf("Analysing left prolongation ( (%s), %s )...\n",
             fAlgToStr( g ), fMonToStr( candidateVariable ) );
    else
      printf("Analysing right prolongation ( (%s), %s )...\n",
             fAlgToStr( g ), fMonToStr( candidateVariable ) );
  }

  // Adjust T − Remove (g, u, P) from T and add (g, u, (enlarged P))
  Tp = fAlgListRemoveNumber( candidatePos, Tp );
  Tp = fAlgListPush( g, Tp );
  Tm = fMonListRemoveNumber( candidatePos, Tm );
  Tm = fMonListPush( u, Tm );
  Tv = fMonPairListRemoveNumber( candidatePos, Tv );

  if( LorR == 0 ) // Left prolongation
    P.lft = multiplicativeUnion( P.lft, candidateVariable );
  else // Right prolongation
    P.rt = multiplicativeUnion( P.rt, candidateVariable );

  Tv = fMonPairListPush( P.lft, P.rt, Tv );

  // Construct the prolongation
```

```
        if( LorR == 0 )
          gDotx = fAlgTimes( fAlgMonom( qOne(), candidateVariable ), g );
        else
          gDotx = fAlgTimes( g, fAlgMonom( qOne(), candidateVariable ) );

        // If the criterion is false...
        // if( NCcriterion( gDotx, u, Tp, Tm, GBasis, vars ) == 0 )
        {
          // ...then find the normal form of the prolongation w.r.t. GBasis
          soFar = fAlgListPush( gDotx, soFar ); // Indicate we have encountered another polynomial
          h = IPolyReduce( gDotx, GBasis, vars ); // Involutively reduce gDotx w.r.t. GBasis
          h = findGCD( h ); // Divide through by the GCD
          if( pl > 2 ) printf("...Reduced␣prolongation␣to␣%s...\n", fAlgToStr( h ) );
          nOfProlongations++; // Increment the number of prolongations processed

          // Check for trivial ideal
          if( fAlgIsOne( h ) == (Bool) 1 ) return fAlgListSingle( fAlgOne() );

          if( fAlgIsZero( h ) == (Bool) 0 ) // If the prolongation did not reduce to 0
          {
            // Add h to GBasis and recalculate multiplicative variables if necessary
            if( IType < 3 )
            {
              pos = 1;
              if( SType == 1 ) GBasis = fAlgListDegRevLexPushPosition( h, GBasis, &pos ); // DegRevLex sort
              else if( SType == 2 ) GBasis = fAlgListAppend( GBasis, fAlgListSingle( h ) ); // Just append
              else GBasis = fAlgListNormalPush( h, GBasis ); // Sort by monomial ordering

              vars = OverlapDiv( GBasis ); // Full recalculate
            }
            else GBasis = fAlgListAppend( GBasis, fAlgListSingle( h ) ); // Just append onto end

            if( pl > 1 ) printf("Added␣%s␣to␣G␣(%u)...\n", fAlgToStr( h ), fAlgListLength( GBasis ) );
            else if( pl == 1 ) printf("Added␣a␣polynomial␣to␣G␣␣␣␣␣␣␣(%u)...\n", fAlgListLength( GBasis ));

            LMh = fAlgLeadMonom( h );

            if( degRestrict == 1 ) // If we are restricting prolongations by degree...
            {
              degTest = fMonLength( LMh );
              if( degTest > d ) // ...and if the degree of the new polynomial exceeds the bound...
              {
                // ...adjust the bound accordingly
                d = degTest;
                if( pl > 2 ) printf("New␣value␣of␣d␣=␣%u\n", d );
                twod = 2*d;
              }
            }

            // if LM(h) == LM(prolongation)
            if( fMonEqual( fAlgLeadMonom( gDotx ), LMh ) == (Bool) 1 )
            {
              // Add entry (h, u, (\emptyset, \emptyset)) to T
```

```
        Tp = fAlgListPush( h, Tp );
        Tm = fMonListPush( u, Tm );
        Tv = fMonPairListPush( fMonOne(), fMonOne(), Tv );
      }
      else // Add entry to T and adjust lists
      {
        // Add entry to T
        Tp = fAlgListPush( h, Tp );
        Tm = fMonListPush( LMh, Tm );
        Tv = fMonPairListPush( fMonOne(), fMonOne(), Tv );
        if( pl > 3 ) printf("Modifying T (size %u)...\n", fAlgListLength( Tp ) );

        // Set up lists for next operation
        Tp2 = fAlgListNul;
        Tm2 = fMonListNul;
        Tv2 = fMonPairListNul;

        // For each (f, v, (DL, DR)) \in T
        if( pl > 4 ) printf("Adjusting Multiplicative Variables...\n");
        while( Tp )
        {
          f = Tp -> first; // Extract a polynomial
          LMf = fAlgLeadMonom( f );

          if( pl > 4 ) printf("Testing (%s, %s)\n", fMonToStr( LMh ), fMonToStr( LMf ) );

          // If LM(h) < LM(f)
          if( theOrdFun( LMh, LMf ) == (Bool) 1 )
          {
            // Add entry to Q
            Qp = fAlgListPush( Tp -> first, Qp );
            Qm = fMonListPush( Tm -> first, Qm );
            Qv = fMonPairListPush( Tv -> lft, Tv -> rt, Qv );

            // Discard f from GBasis
            GBasis = fAlgListFXRem( GBasis, f );
            if( pl > 1 ) printf("Discarded %s from G (%u)...\n", fAlgToStr( f ), fAlgListLength( GBasis ) );
            else if( pl == 1 ) printf("Discarded a polynomial from G (%u)...\n", fAlgListLength( GBasis ) );
          }
          else
          {
            // Keep entry in T
            Tp2 = fAlgListPush( Tp -> first, Tp2 );
            Tm2 = fMonListPush( Tm -> first, Tm2 );
            Tv2 = fMonPairListPush( Tv -> lft, Tv -> rt, Tv2 );
          }
          // Advance the lists to the next entry
          Tp = Tp -> rest;
          Tm = Tm -> rest;
          Tv = Tv -> rest;
        }

        // Set up lists for next operation
```

```
        Tp = fAlgListNul;
        Tm = fMonListNul;
        Tv = fMonPairListNul;

        // Recalculate multiplicative variables
        if( IType < 3 ) vars = OverlapDiv( GBasis );

        // For each (f, v, (DL, DR)) \in T
        while( Tp2 )
        {
          // Keep f and v as they are
          f = Tp2 -> first;
          Tp = fAlgListPush( f, Tp );
          Tm = fMonListPush( Tm2 -> first, Tm );
          DL = Tv2 -> lft;
          DR = Tv2 -> rt;

          // Find intersection of D and NM_I(f, GBasis)
          if( IType < 3 ) // Local division
          {
            // Find NM_I(f, GBasis)
            pos = fAlgListPosition( f, GBasis );
            fVars = fMonPairListNumber( pos, vars );
            fVarsL = fVars.lft;
            fVarsR = fVars.rt;

            NML = fMonOne();
            NMR = fMonOne();
            j = 1;

            // Calculate the intersection
            while( j <= (ULong) nOfGenerators )
            {
              gen = ASCIIMon( j );

              // If gen appears in DL (nonmultiplicatives) but not in fVarsL (multiplicatives)
              if ( ( fMonIsMultiplicative( gen, DL ) == 1 )
                   && ( fMonIsMultiplicative( gen, fVarsL ) == 0 ) )
                NML = fMonTimes( NML, gen ); // gen appears in the left intersection
              // If gen appears in DR (nonmultiplicatives) but not in fVarsR (multiplicatives)
              if ( ( fMonIsMultiplicative( gen, DR ) == 1 )
                   && ( fMonIsMultiplicative( gen, fVarsR ) == 0 ) )
                NMR = fMonTimes( NMR, gen ); // gen appears in the right intersection

              j++; // Get ready to look at the next variable
            }
          }
          else if ( IType >= 3 ) // Global division
          {
            // Find the multiplicative variables
            if( IType == 3 ) LMultVars( fAlgLeadMonom( f ), &cutoffL, &cutoffR );
            else if( IType == 4 ) RMultVars( fAlgLeadMonom( f ), &cutoffL, &cutoffR );
            else EMultVars( fAlgLeadMonom( f ), &cutoffL, &cutoffR );
```

```
                  NML = fMonOne();
                  NMR = fMonOne();

                  // Calculate the left intersection
                  for( j = cutoffR+1; j <= (ULong) nOfGenerators; j++ )
                  {
                    gen = ASCIIMon( j ); // Obtain a nonmultiplicative variable
                    // If it appears in DL it appears in the intersection
                    if( fMonIsMultiplicative( gen, DL ) == 1 )
                      NML = fMonTimes( NML, gen );
                  }

                  // Calculate the right intersection
                  for( j = 1; j < cutoffL; j++ )
                  {
                    gen = ASCIIMon( j ); // Obtain a nonmultiplicative variable
                    // If it appears in DR it appears in the intersection
                    if( fMonIsMultiplicative( gen, DR ) == 1 )
                      NMR = fMonTimes( NMR, gen );
                  }
                }

                // Add the nonmultiplicative variables to Tv
                Tv = fMonPairListPush( NML, NMR, Tv );

                // Advance the lists
                Tp2 = Tp2 -> rest;
                Tm2 = Tm2 -> rest;
                Tv2 = Tv2 -> rest;
              }
            }
          }
        }
        // else if( pl > 2 ) printf("...Criterion used to discard prolongation...\n");
      }
      else // exit from loop - no suitable prolongations found
      {
        escape = 1;
      }
    }
  }
  while( Qp );

  if( pl > 0 ) printf("...Involutive␣Basis␣Computed.\n");

  return GBasis;
}

/*
 * ===========
 * End of File
 * ===========
 */
```

B.2.12 involutive.c

```
/*
 * File: involutive.c (Noncommutative Involutive Basis Program)
 * Author: Gareth Evans
 * Last Modified: 10th August 2005
 */

// Include MSSRC Libraries
# include <fralg.h>

// Include *_functions Libraries
# include "file_functions.h"
# include "list_functions.h"
# include "fralg_functions.h"
# include "arithmetic_functions.h"
# include "ncinv_functions.h"

/*
 * ===========================================
 * External Variables for ncinv_functions.c
 * ===========================================
 */

ULong nOfProlongations; // Stores the number of prolongations calculated
int degRestrict = 0, // Determines whether of not prolongations are restricted by degree
    IType = 3, // Stores the involutive division used (1,2 = Left/Right Overlap, 3,4, = Left/Right, 5 = Empty)
    EType = 0, // Stores the type of Overlap Division
    SType = 1, // Determines how the basis is sorted
    MType = 1; // Determines method of involutive division

/*
 * ===================================================================
 * External Variables for fralg_functions.c AND ncinv_functions.c
 * ===================================================================
 */

ULong nRed = 0; // Stores how many reductions have been carried out
int nOfGenerators, // Holds the number of generators
    pl = 1; // Holds the "Print Level"

/*
 * =========================================
 * Global Variables for ncinv_functions.c
 * =========================================
 */

FMonList gens = fMonListNul; // Stores the generators for the basis
FMonPairList multVars = fMonPairListNul; // Stores multiplicative variables
FAlgList F = fAlgListNul, // Holds the input basis
         G = fAlgListNul, // Holds the Groebner Basis
         G_Reduced = fAlgListNul, // Holds the Reduced Groebner Basis
         IB = fAlgListNul, // Holds the Involutive Basis
```

```
          IMPChecker = fAlgListNul; // Stores a list of polynomials for the IMP
FMon allVars; // Stores all the variables
int AlgType = 1, // Stores which involutive algorithm to use
    order_switch = 1; // Stores the monomial ordering used

/*
 * Remark: Here are the possible values of order_switch:
 * 1: DegRevLex
 * 2: DegLex
 * 3: Lex
 * 9: Wreath Product
 */

/*
 * Function Name: NormalBatch
 *
 * Overview: Calculates an Involutive Basis and a
 * Reduced Minimal Groebner Basis
 *
 * Detail: Given an input basis, this function uses the
 * functions in fralg_functions.c and ncinv_functions.c
 * to calculate an Involutive Basis and a minimal
 * reduced Groebner Basis for the input basis.
 *
 * External Variables Used: int pl;
 * Global Variables Used: FAlgList F, G, G_Reduced;
 *
 */
static void
NormalBatch( )
{
  FAlgList Display = fAlgListNul;
  int plSwap = pl;

  // Output some initial information to screen
  if( pl > 0 )
  {
    printf("\nPolynomials in the input basis:\n");
    Display = fAlgListCopy( F );
    while( Display )
    {
      // If pl == 1, display the polynomial using the original generators
      if( pl == 1 ) printf("%s,\n", postProcess( Display -> first, gens ) );
      // Otherwise, if pl > 1, display the polynomial using ASCII generators
      else if( pl > 1 ) printf("%s,\n", fAlgToStr( Display -> first ) );
      Display = Display -> rest; // Advance the list
    }
    printf("[%u Polynomials]\n", fAlgListLength( F ) );
  }

  // Calculate an Involutive Basis for F
  if( AlgType == 1 ) G = Gerdt( F );
  else G = Seiler( F );
```

```

// Display calculated basis
if( pl > 0 )
{
  if( pl > 1 ) printf("Number of Prolongations Considered = %u\n", nOfProlongations );
  if( IType < 3 ) // Local division
  {
    printf("\nHere is the Involutive Basis\n((Left, Right) Multiplicative Variables in Brackets):\n");
    IB = fAlgListCopy( G );
    Display = fAlgListCopy( G );

    // We will now calculate the multiplicative variables silently
    pl = 0; // Set silent print level
    if( IType < 3 ) multVars = OverlapDiv( G );
    pl = plSwap; // Restore original print level

    while( Display )
    {
      // If pl == 1, display the polynomial using the original generators
      if( pl == 1 ) printf("%s, (%s, %s),\n", postProcess( Display -> first, gens ),
                           postProcess( fAlgMonom( qOne(), fMonReverse( multVars -> lft ) ), gens ),
                           postProcess( fAlgMonom( qOne(), fMonReverse( multVars -> rt ) ), gens ) );
      // Otherwise, if pl > 1, display the polynomial using ASCII generators
      else if( pl > 1 ) printf("%s, (%s, %s),\n", fAlgToStr( Display -> first ),
                               fMonToStr( fMonReverse( multVars -> lft ) ),
                               fMonToStr( fMonReverse( multVars -> rt ) ) );
      Display = Display -> rest; // Advance the polynomial list
      multVars = multVars -> rest; // Advance the multiplicative variables list
    }
    printf("[%u Polynomials]\n", fAlgListLength( G ) );
  }
  else // Global division
  {
    printf("\nHere is the Involutive Basis\n((Left, Right) Multiplicative Variables in Brackets):\n");
    IB = fAlgListCopy( G );
    Display = fAlgListCopy( G );
    while( Display )
    {
      if( IType == 3 ) // Left Division
      {
        // If pl == 1, display the polynomial using the original generators
        if( pl == 1 ) printf("%s, (%s, 1),\n", postProcess( Display -> first, gens ), fMonToStr( allVars ) );
        // Otherwise, if pl > 1, display the polynomial using ASCII generators
        else if( pl > 1 ) printf("%s, (all, none),\n", fAlgToStr( Display -> first ) );
      }
      else if( IType == 4 ) // Right Division
      {
        // If pl == 1, display the polynomial using the original generators
        if( pl == 1 ) printf("%s, (1, %s),\n", postProcess( Display -> first, gens ), fMonToStr( allVars ) );
        // Otherwise, if pl > 1, display the polynomial using ASCII generators
        else if( pl > 1 ) printf("%s, (none, all),\n", fAlgToStr( Display -> first ) );
      }
      else if( IType == 5 ) // Empty Division
```

```
        {
          // If pl == 1, display the polynomial using the original generators
          if( pl == 1 ) printf("%s, (1, 1),\n", postProcess( Display -> first, gens ) );
          // Otherwise, if pl > 1, display the polynomial using ASCII generators
          else if( pl > 1 ) printf("%s, (none, none),\n", fAlgToStr( Display -> first ) );
        }
        Display = Display -> rest; // Advance the list
      }
      printf("[%u Polynomials]\n", fAlgListLength( G ) );
    }
  }

  // Calculate a reduced and minimal Groebner Basis
  if( pl > 0 ) printf("\nComputing the Reduced Groebner Basis...\n");
  G = minimalGB( G ); // Minimise the basis
  G_Reduced = reducedGB( G ); // Reduce the basis
  if( pl > 0 ) printf("...Reduced Groebner Basis Computed.\n");

  // Display some information on screen
  if( pl > 0 )
  {
    printf("\nHere is the Reduced Groebner Basis:\n");
    Display = fAlgListCopy( G_Reduced );
    while( Display )
    {
      // If pl == 1, display the polynomial using the original generators
      if( pl == 1 ) printf("%s,\n", postProcess( Display -> first, gens ) );
      // Otherwise, if pl > 1, display the polynomial using ASCII generators
      else if( pl > 1 ) printf("%s,\n", fAlgToStr( Display -> first ) );
      Display = Display -> rest;
    }
    printf("[%u Polynomials]\n", fAlgListLength( G_Reduced ) );
  }
}

/*
 * Function Name: IMPSolver
 *
 * Overview: Solves the Ideal Membership Problem for polynomials
 * sourced from disk or from user input
 *
 * Detail: Given a polynomial sourced from disk or from user
 * input, this function solves the ideal membership problem
 * for that polynomial by reducing the polynomial w.r.t.
 * a minimal reduced Groebner Basis (using a specially
 * adapted function) and testing to see whether the
 * polynomial reduces to zero or not.
 *
 * External Variables Used: FAlgList IMPChecker;
 * FMonList gens;
 * int pl;
 */
static void
```

```
IMPSolver( )
{
  FAlgList polynomials = fAlgListNul;
  FAlg polynomial;
  int sink;
  Short dk = 2; // Convention: 1 = disk, 2 = keyboard
  Bool answer;
  String inputChar = strNew(), inputStr = strNew(),
         polyFileName = strNew(), outputString = strNew();
  FILE *polyFile;

  // Determine whether the input will come from disk or from the keyboard
  printf("*** IDEAL MEMBERSHIP PROBLEM SOLVER ***\n\n");
  printf("Source: Disk (d) or Keyboard (k)? ...");
  sink = scanf( "%s", inputChar );

  // If the user hasn't entered 'd' or 'k', ask for another letter
  while( ( strEqual( inputChar, "d" ) == 0 ) & ( strEqual( inputChar, "k" ) == 0 ) )
  {
    printf("Error: Please enter d or k ...");
    sink = scanf( "%s", inputChar );
  }
  printf("\n");

  // If the polynomials are to be obtained from disk
  if( strEqual( inputChar, "d" ) == (Bool) 1 )
  {
    dk = 1; // Set input from disk
    printf("Please enter the file name of the input polynomials ...");
    sink = scanf( "%s", polyFileName );

    // Read file from disk
    if( ( polyFile = fopen( polyFileName, "r" ) ) == NULL )
    {
      printf("%s\n", "Error opening the polynomial input file.");
      exit( EXIT_FAILURE );
    }

    // Obtain the polynomials from the file
    polynomials = fAlgListFromFile( polyFile );
    polynomials = preProcess( polynomials, gens ); // Change to ASCII order
    sink = fclose( polyFile );
  }
  else // Else obtain the first polynomial from the keyboard
  {
    if( pl < 2 ) // Require polynomial using original generators
      printf("Please enter a polynomial (e.g. x*y^2-z)\n");
    else // Require polynomial using ASCII generators
      printf("Please enter a polynomial (e.g. AAA*AAB^2-AAC)\n");
    printf("(A semicolon terminates the program)...");
    sink = scanf( "%s", inputStr );

    if( ( strEqual( inputStr, "" ) == (Bool) 1 ) | ( strEqual( inputStr, ";" ) == (Bool) 1 ) )
```

```c
      polynomials = fAlgListNul; // No poly given, terminate program
    else
    {
      // Push the given polynomial onto the list
      polynomials = fAlgListPush( parseStrToFAlg( inputStr ), polynomials );
      if( pl < 2 ) // Need to convert to ASCII order
        polynomials = preProcess( polynomials, gens );
    }
  }

  // For each polynomial in the list (for keyboard entry the list will have 1 element)
  while( polynomials )
  {
    polynomial = polynomials -> first; // Extract a polynomial to test
    polynomials = polynomials -> rest; // Advance the list

    // Solve the Ideal Membership Problem for the polynomial
    // using the Groebner Basis stored in IMPChecker
    answer = idealMembershipProblem( polynomial, IMPChecker );

    // Prepare to report the result correctly
    if( pl < 2 ) outputString = postProcess( polynomial, gens );
    else outputString = fAlgToStr( polynomial );

    // Return the results
    if( answer == (Bool) 0 )
      printf("Polynomial %s is NOT a member of the ideal.\n", outputString );
    else
      printf("Polynomial %s IS a member of the ideal.\n", outputString );

    if( dk == 2 ) // Obtain another poly from keyboard
    {
      if( pl < 2 ) // Require polynomial using original generators
        printf("Please enter a polynomial (e.g. x*y^2-z)\n");
      else // Require polynomial using ASCII generators
        printf("Please enter a polynomial (e.g. AAA*AAB^2-AAC)\n");
      printf("(A semicolon terminates the program)...");
      sink = scanf( "%s", inputStr );

      if( ( strEqual( inputStr, "" ) == (Bool) 1 ) | ( strEqual( inputStr, ";" ) == (Bool) 1 ) )
        polynomials = fAlgListNul; // No poly given, terminate program
      else
      {
        // Push the given polynomial onto the list
        polynomials = fAlgListPush( parseStrToFAlg( inputStr ), polynomials );
        if( pl < 2 ) // Need to convert to ASCII order
          polynomials = preProcess( polynomials, gens );
      }
    }
  }
}

/*
```

```c
 * Function Name: main
 *
 * Overview: A Noncommutative Involutive Basis Program
 *
 * Detail: This function deals with the inputs and outputs
 * of the program. In particular, the command line arguments are
 * processed, the input files are read, and once the Involutive
 * Basis has been calculated, it is output to disk together with
 * the reduced minimal Groebner Basis.
 *
 * External Variables Used: int nOfGenerators, pl;
 * Global Variables Used: FAlgList F;
 * FMonList gens;
 * int order_switch;
 */
int
main( argc, argv )
int argc;
char *argv[];
{
  String filename = strNew(), // Used to create the output file name
         filename2 = strNew(); // Used to create the involutive output file name
  FAlg zeroOrOne; // Used to test for trivial basis elements
  FMonList gens_copy = fMonListNul; // Holds a copy of the generators
  ULong k; // Used as a counter
  int i, // Used as a counter
      length; // Used to store the length of a command line argument
  Short alpha_switch = 0, // Do we optimise the generator order lexicographically?
        fractions = 0, // Do we eliminate fractions from the input basis?
        IMP = 0, // At the end of the algorithm, do we solve the IMP?
        p; // Used to navigate through the command line arguments
  FILE *grobdata, // Stores the input file
       *outputdata; // Used to construct the output file

  // Process Command Line Arguments
  if( argc < 2 )
  {
    printf("\nInvalid Input - wrong number of parameters.");
    printf("\nSee README for more information.\n\n");
    exit( EXIT_FAILURE );
  }

  p = 1; // p will step through all the command line arguments
  while( argv[p][0] == '-' ) // While there is another command line argument
  {
    length = (int) strlen( argv[p] ); // Determine length of argument
    if( pl > 8 ) printf("Looking at parameter %i of length %i\n", p, length );

    if( length == 1 ) // Just a "-" was given
    {
      printf("\nInvalid Input - empty parameter (position %i).", p);
      printf("\nSee README for more information.\n\n");
      exit( EXIT_FAILURE );
```

```
}

// We will now deal with the different allowable parameters
switch( argv[p][1] )
{
  case 'a':
    alpha_switch = 1; // Optimise the generator order lexicographically
    break;
  case 'c': // Choose the algorithm used to construct the involutive basis
    if( length != 3 )
    {
      printf("\nInvalid Input - incorrect length on code parameter.");
      printf("\nSee README for more information.\n\n");
      exit( EXIT_FAILURE );
    }
    switch( argv[p][2] ) // Choose the algorithm type
    {
      case '1' :
      case '2' :
        AlgType = ( (int) argv[p][2] ) - 48;
        break;
      default:
        printf("\nInvalid Parameter (%c is an invalid code selection character).", argv[p][2]);
        printf("\nSee README for more information.\n\n");
        exit( EXIT_FAILURE );
        break;
    }
    break;
  case 'd':
    order_switch = 2; // Use the DegLex Monomial Ordering
    break;
  case 'e': // Choose the Overlap Division type
    if( length != 3 )
    {
      printf("\nInvalid Input - incorrect length on type of Overlap Division parameter.");
      printf("\nSee README for more information.\n\n");
      exit( EXIT_FAILURE );
    }
    switch( argv[p][2] ) // Assign the type
    {
      case '1' :
      case '2' :
      case '3' :
      case '4' :
      case '5' :
        EType = ( (int) argv[p][2] ) - 48;
        break;
      default:
        printf("\nInvalid Parameter (%c is an invalid -e character).", argv[p][2]);
        printf("\nSee README for more information.\n\n");
        exit( EXIT_FAILURE );
        break;
    }
```

```
      break;
    case 'f':
      fractions = 1; // Eliminate fractions from the input basis
      break;
    case 'l':
      order_switch = 3; // Use the Lexicographic Monomial Ordering
      break;
    case 'm': // Choose method of involutive division
      if( length != 3 )
      {
        printf("\nInvalid Input - incorrect length on method parameter.");
        printf("\nSee README for more information.\n\n");
        exit( EXIT_FAILURE );
      }
      switch( argv[p][2] ) // Choose the method
      {
        case '1' :
        case '2' :
          MType = ( (int) argv[p][2] ) - 48;
          break;
        default:
          printf("\nInvalid Parameter (%c is an invalid method character).", argv[p][2]);
          printf("\nSee README for more information.\n\n");
          exit( EXIT_FAILURE );
          break;
      }
      break;
    case 'o': // Choose how the basis is stored
      if( length != 3 )
      {
        printf("\nInvalid Input - incorrect length on sort parameter.");
        printf("\nSee README for more information.\n\n");
        exit( EXIT_FAILURE );
      }
      switch( argv[p][2] ) // Choose the sorting method
      {
        case '1' :
        case '2' :
        case '3' :
          SType = ( (int) argv[p][2] ) - 48;
          break;
        default:
          printf("\nInvalid Parameter (%c is an invalid sort character).", argv[p][2]);
          printf("\nSee README for more information.\n\n");
          exit( EXIT_SUCCESS );
          break;
      }
      break;
    case 'p': // Calls the Interactive Ideal Membership Problem
      IMP = 1; // Solver after the Groebner Basis has been found.
      break;
    case 'r': // Use the DegRevLex Monomial Ordering
      break; // (we do nothing here - this is default option)
```

```
      case 's': // Choose an involutive division
        if( length != 3 )
        {
          printf("\nInvalid Input - incorrect length on selection parameter.");
          printf("\nSee README for more information.\n\n");
          exit( EXIT_SUCCESS );
        }
        switch( argv[p][2] ) // Assign the involutive division type
        {
          case '1' :
          case '2' :
          case '3' :
          case '4' :
          case '5' :
            IType = ( (int) argv[p][2] ) - 48;
            break;
          default:
            printf("\nInvalid Parameter (%c is an invalid involutive division character).", argv[p][2]);
            printf("\nSee README for more information.\n\n");
            exit( EXIT_FAILURE );
            break;
        }
        break;
      case 'v': // Choose the amount of information given to screen
        if( length != 3 )
        {
          printf("\nInvalid Input - incorrect length on verbose parameter.");
          printf("\nSee README for more information.\n\n");
          exit( EXIT_FAILURE );
        }
        switch( argv[p][2] )
        {
          case '0' :
          case '1' :
          case '2' :
          case '3' :
          case '4' :
          case '5' :
          case '6' :
          case '7' :
          case '8' :
          case '9' :
            pl = ( (int) argv[p][2] ) - 48;
            break;
          default:
            printf("\nInvalid Parameters (%c is an invalid verbose character).", argv[p][2]);
            printf("\nSee README for more information.\n\n");
            exit( EXIT_FAILURE );
            break;
        }
        break;
      case 'w':
        order_switch = 9; // Use the Wreath Product Monomial Ordering
```

```c
        break;
      case 'x':
        degRestrict = 1; // Turns on restriction of prolongations by degree
        break;
      default:
        printf("\nInvalid Parameter (%c is an invalid character).", argv[p][1]);
        printf("\nSee README for more information.\n\n");
        exit( EXIT_FAILURE );
        break;
    }
    p++; // Get ready to look at the next parameter
  }

  p = p-1; // p now holds the number of parameters processed

  // Test overloading of switches
  if( filenameLength( argv[1+p] ) > 59 )
  {
    printf("\nError: The input filename must not\n");
    printf("exceed 59 characters. Exiting...\n\n");
    exit( EXIT_SUCCESS );
  }

  if( ( EType > 0 ) && ( IType >= 3 ) )
  {
    printf("\nError: The -e(n) option must be used with\n");
    printf("either the -s1 or -s2 options. Exiting...\n\n");
    exit( EXIT_SUCCESS );
  }

  if( ( EType == 2 ) && ( MType == 1 ) )
  {
    printf("\n***   Warning: The Selected Overlap Division Type is not a    ***\n");
      printf("*** strong involutive division when used with the -m1 option. ***\n");
  }

  // Open file specified on the command line
  if( ( grobdata = fopen ( argv[1+p], "r" ) ) == NULL )
  {
    printf("Error opening the input file.\n");
    exit( EXIT_FAILURE );
  }

  /*
   * The first line of the input file should contain the
   * generators in the format a; b; c; ...
   * (representing a > b > c > ...). We will now read the
   * generators from file and calculate the number of
   * generators obtained.
   */
  gens = fMonListFromFile( grobdata );

  /*
```

```c
 * As the rest of the program assumes a generator order
 * a < b < c < ... (for ASCII comparison), we now reverse
 * the list of generators.
 */
gens = fMonListFXRev( gens );

k = fMonListLength( gens );
if( k >= (ULong) INT_MAX ) // Check limit
{
  printf("Error: INT_MAX Exceeded (in main)\n");
  exit( EXIT_FAILURE );
}
else nOfGenerators = (int) k;

// Check generator bound
if( nOfGenerators > 17576 )
{
  printf("Error: The number of generators must not exceed 17576\n");
  exit( EXIT_FAILURE );
}

if( IType >= 3 ) // Global division
{
  // Create a monomial storing all the generators in order
  gens_copy = fMonListCopy( gens );
  allVars = fMonOne();
  while( gens_copy )
  {
    allVars = fMonTimes( allVars, gens_copy -> first );
    gens_copy = gens_copy -> rest;
  }
  allVars = fMonReverse( allVars );
}

// Welcome
if( pl > 0 )
{
  if( IType < 3 ) printf("\n*** NONCOMMUTATIVE INVOLUTIVE BASIS PROGRAM (LOCAL DIVISION) *** \n");
  else printf("\n*** NONCOMMUTATIVE INVOLUTIVE BASIS PROGRAM (GLOBAL DIVISION) *** \n");
}

// We will now choose the monomial ordering to be used.
switch( order_switch )
{
  case 1:
    theOrdFun = fMonDegRevLex;
    if( pl > 0 ) printf("\nUsing the DegRevLex Ordering with ");
    break;
  case 2:
    theOrdFun = fMonTLex;
    if( pl > 0 ) printf("\nUsing the DegLex Ordering with ");
    break;
  case 3:
```

```
      theOrdFun = fMonLex;
      if( pl > 0 ) printf("\nUsing the Lex Ordering with ");
      break;
    case 9:
      theOrdFun = fMonWreathProd;
      if( pl > 0 ) printf("\nUsing the Wreath Product Ordering with ");
      break;
    default:
      break;
  }

  // Output the generator order to screen...
  if( pl > 0 )
  {
    fMonListDisplayOrder( gens );
    printf("\n");
  }

  // Now read the polynomials from disk
  F = fAlgListFromFile( grobdata );

  // If necessary, optimise the generator order
  if( alpha_switch == 1 ) gens = alphabetOptimise( gens, F );

  /*
   * Now substitute original generators for ASCII generators in all
   * basis polynomials. This is done because all the monomial
   * orderings use ASCII string comparisons for efficiency.
   * For example, if the original monomial ordering is x > y > z
   * and a polynomial x*y-2*z is in the basis, then the polynomial
   * we get after substituting for the ASCII order (AAC > AAB > AAA) is
   * AAC*AAB-2*AAA.
   */
  G = preProcess( F, gens ); // Note: placed in G for processing
  F = fAlgListNul;

  // If we are asked to remove all fractions from the input basis, do so now.
  if( fractions == 1 ) G = fAlgListRemoveFractions( G );

  // Test the list for special cases (trivial ideals)
  while( G )
  {
    zeroOrOne = G -> first; // Extract a polynomial
    if( fAlgIsZero( zeroOrOne ) == (Bool) 0 ) // If the polynomial is not equal to 0...
      F = fAlgListPush( zeroOrOne, F ); // ...add to the input list
    // Now divide by the leading coefficient to get a unit coefficient
    zeroOrOne = fAlgScaDiv( zeroOrOne, fAlgLeadCoef( zeroOrOne ) );
    if( fAlgIsOne( zeroOrOne ) == (Bool) 1 ) // If the polynomial is equal to 1...
    {
      // ... we have a trivial ideal
      F = fAlgListSingle( fAlgOne() );
      break;
    }
```

```
  G = G -> rest; // Advance the list
}
F = fAlgListFXRev( F ); // Reverse the list (it was constructed in reverse)

G = fAlgListNul; // Reset for later use

// Calculate the number of polynomials in the input basis
k = fAlgListLength( F );
if( k >= (ULong) INT_MAX ) // Check limit
{
  printf("Error: INT_MAX Exceeded (in main)\n");
  exit( EXIT_FAILURE );
}

// Calculate an Involutive Basis for F followed by a
// reduced and minimal Groebner Basis for F
NormalBatch();

// Write Reduced Groebner Basis to Disk
if( pl > 0 ) printf("\nWriting Reduced Groebner Basis to Disk... ");

// Choose the correct suffix for the filename (argv[1+p] is the original filename)
switch( order_switch )
{
  case 1:
    filename = appendDotDegRevLex( argv[1+p] );
    break;
  case 2:
    filename = appendDotDegLex( argv[1+p] );
    break;
  case 3:
    filename = appendDotLex( argv[1+p] );
    break;
  case 9:
    filename = appendDotWP( argv[1+p] );
    break;
  default:
    printf("\nERROR DURING SUFFIX SELECTION\n\n");
    exit( EXIT_FAILURE );
    break;
}
filename2 = strConcat( filename, ".inv" );

// Now open the output file
if( ( outputdata = fopen ( filename, "w" ) ) == NULL )
{
  printf("%s\n", "Error opening / creating the (first) output file.");
  exit( EXIT_FAILURE );
}

// Write the (reversed) generator order to disk
fMonListToFile( outputdata, fMonListRev( gens ) );
```

```
// Write Polynomials to disk
G = fAlgListNul;

// If we are required to solve the Ideal Membership Problem,
// let us make a copy of the output basis now
if( IMP == 1 ) IMPChecker = fAlgListCopy( G_Reduced );

// We will now convert all polynomials in the basis
// from ASCII order back to the user's order, writing
// the converted polynomials to file as we go.
while( G_Reduced )
{
  fprintf( outputdata, "%s;\n", postProcessParse( G_Reduced -> first, gens ) );
  G_Reduced = G_Reduced -> rest;
}

// Close off the output file
i = fclose( outputdata );

if( pl > 0 ) printf("Done.\nWriting Involutive Basis to Disk... ");

// Now write the Involutive Basis to disk
if( ( outputdata = fopen ( filename2, "w" ) ) == NULL )
{
  printf("%s\n", "Error opening / creating the (second) output file.");
  exit( EXIT_FAILURE );
}

// Write the (reversed) generator order to disk
fMonListToFile( outputdata, fMonListRev( gens ) );

// If we are using a local division we need to find the multiplicative variables now
if( IType < 3 ) multVars = OverlapDiv( IB );

while( IB )
{
  fprintf( outputdata, "%s; ", postProcessParse( IB -> first, gens ) );
  if( IType < 3 ) // Overlap-based Division
  {
    fprintf( outputdata, "(%s, %s);\n",
             postProcess( fAlgMonom( qOne(), fMonReverse( multVars -> lft ) ), gens ),
             postProcess( fAlgMonom( qOne(), fMonReverse( multVars -> rt ) ), gens ) );
  }
  else if( IType == 3 ) // Left Division
  {
    fprintf( outputdata, "(%s, 1);\n", fMonToStr( allVars ) );
  }
  else if( IType == 4 ) // Right Division
  {
    fprintf( outputdata, "(1, %s);\n", fMonToStr( allVars ) );
  }
  else if( IType == 5 ) // Empty Division
  {
```

```
      fprintf( outputdata, "(1, 1);\n" );
    }

    IB = IB -> rest; // Advance the list of rules
    // If need be, advance the multiplicative variables list
    if( IType < 3 ) multVars = multVars -> rest;
  }

  // Close off the output file
  i = fclose( outputdata );

  if( pl > 0 ) printf("Done.\n\n");

  // If the Ideal Membership Problem Solver is required, run it now.
  if( IMP == 1 ) IMPSolver();

  return EXIT_SUCCESS; // Exit successfully
}

# include "file_functions.c"
# include "list_functions.c"
# include "fralg_functions.c"
# include "arithmetic_functions.c"
# include "ncinv_functions.c"

// End of File
```

Appendix C

Program Output

In this Appendix, we provide sample sessions showing how the program given in Appendix B can be used to compute noncommutative Involutive Bases with respect to different involutive divisions and monomial orderings.

C.1 Sample Sessions

C.1.1 Session 1: Locally Involutive Bases

Task: If $F := \{x^2y^2 - 2xy^2 + x^2,\ x^2y - 2xy\}$ generates an ideal J over the polynomial ring $\mathbb{Q}\langle x, y\rangle$, compute a Locally Involutive Basis for F with respect to the strong left overlap division $\mathcal{S}$; thick divisors; and the DegLex monomial ordering.

Origin of Example: Example 5.7.1.

Input File:

```
x; y;
x^2*y^2 - 2*x*y^2 + x^2;
x^2*y - 2*x*y;
```

Plan: Apply the program given in Appendix B to the above file, using the '-c2' option to select Algorithm 12; the '-d' option to select the DegLex monomial ordering; the '-m2' option to select thick divisors; and the '-e2' and '-s1' options to select the strong left overlap division.

Program Output:

```
ma6:mssrc−aux/thesis> time involutive −c2 −d −e2 −m2 −s1 thesis1.in

*** NONCOMMUTATIVE INVOLUTIVE BASIS PROGRAM (LOCAL DIVISION) ***

Using the DegLex Ordering with x > y

Polynomials in the input basis:
x^2 y^2 − 2 x y^2 + x^2,
x^2 y − 2 x y,
[2 Polynomials]

Computing an Involutive Basis...
Added Polynomial #3 to Basis...
Added Polynomial #4 to Basis...
Autoreduction reduced the basis to size 3...
Added Polynomial #4 to Basis...
Autoreduction reduced the basis to size 3...
Added Polynomial #4 to Basis...
Added Polynomial #5 to Basis...
...Involutive Basis Computed.

Here is the Involutive Basis
((Left, Right) Multiplicative Variables in Brackets):
x y^2 x, (x y, 1),
x y^2, (x y, y),
x y x, (x y, 1),
x y, (x y, 1),
x^2, (x y, 1),
[5 Polynomials]

Computing the Reduced Groebner Basis...
...Reduced Groebner Basis Computed.

Here is the Reduced Groebner Basis:
x y,
x^2,
[2 Polynomials]

Writing Reduced Groebner Basis to Disk... Done.
Writing Involutive Basis to Disk... Done.

0.000u 0.007s 0:00.15 0.0% 0+0k 0+2io 16pf+0w
ma6:mssrc−aux/thesis>
```

Output File:

```
x; y;
x*y^2*x; (x y, 1);
x*y^2; (x y, y);
x*y*x; (x y, 1);
x*y; (x y, 1);
```

```
x^2; (x y, 1);
```

C.1.2 Session 2: Involutive Complete Rewrite Systems

Task: If $F := \{x^3 - 1,\ y^2 - 1,\ (xy)^2 - 1,\ Xx - 1,\ xX - 1,\ Yy - 1,\ yY - 1\}$ generates an ideal J over the polynomial ring $\mathbb{Q}\langle Y, X, y, x\rangle$, compute an Involutive Basis for F with respect to the left division $\triangleleft$ and the DegLex monomial ordering.

Origin of Example: Example 5.7.3 (F corresponds to a monoid rewrite system for the group S_3; we want to compute an involutive complete rewrite system for S_3).

Input File:

```
Y; X; y; x;
x^3 - 1;
y^2 - 1;
(x*y)^2 - 1;
X*x - 1;
x*X - 1;
Y*y - 1;
y*Y - 1;
```

Plan: Apply the program given in Appendix B to the above file, using the '-c2' option to select Algorithm 12 and the '-d' option to select the DegLex monomial ordering (the left division is selected by default).

Program Output:

```
ma6:mssrc-aux/thesis> time involutive -c2 -d thesis2.in

*** NONCOMMUTATIVE INVOLUTIVE BASIS PROGRAM (GLOBAL DIVISION) ***

Using the DegLex Ordering with Y > X > y > x

Polynomials in the input basis:
x^3 - 1,
y^2 - 1,
x y x y - 1,
X x - 1,
x X - 1,
Y y - 1,
y Y - 1,
[7 Polynomials]

Computing an Involutive Basis...
Added Polynomial #8 to Basis...
Added Polynomial #9 to Basis...
```

```
Added Polynomial #10 to Basis...
Added Polynomial #11 to Basis...
Added Polynomial #12 to Basis...
Added Polynomial #13 to Basis...
Autoreduction reduced the basis to size 11...
Added Polynomial #12 to Basis...
Added Polynomial #13 to Basis...
Added Polynomial #14 to Basis...
Added Polynomial #15 to Basis...
Added Polynomial #16 to Basis...
Added Polynomial #17 to Basis...
Added Polynomial #18 to Basis...
Added Polynomial #19 to Basis...
Added Polynomial #20 to Basis...
Added Polynomial #21 to Basis...
Added Polynomial #22 to Basis...
Added Polynomial #23 to Basis...
Autoreduction reduced the basis to size 19...
Added Polynomial #20 to Basis...
Autoreduction reduced the basis to size 19...
Added Polynomial #20 to Basis...
Added Polynomial #21 to Basis...
Added Polynomial #22 to Basis...
Added Polynomial #23 to Basis...
Added Polynomial #24 to Basis...
Added Polynomial #25 to Basis...
Added Polynomial #26 to Basis...
Added Polynomial #27 to Basis...
Added Polynomial #28 to Basis...
Added Polynomial #29 to Basis...
Added Polynomial #30 to Basis...
Added Polynomial #31 to Basis...
Added Polynomial #32 to Basis...
Added Polynomial #33 to Basis...
Added Polynomial #34 to Basis...
Added Polynomial #35 to Basis...
Added Polynomial #36 to Basis...
Added Polynomial #37 to Basis...
Added Polynomial #38 to Basis...
Added Polynomial #39 to Basis...
Added Polynomial #40 to Basis...
Autoreduction reduced the basis to size 29...
Added Polynomial #30 to Basis...
Autoreduction reduced the basis to size 19...
...Involutive Basis Computed.

Here is the Involutive Basis
((Left, Right) Multiplicative Variables in Brackets):
y^2 - 1, (Y X y x, 1),
X x - 1, (Y X y x, 1),
x X - 1, (Y X y x, 1),
Y y - 1, (Y X y x, 1),
y^2 x - x, (Y X y x, 1),
```

```
Y − y, (Y X y x, 1),
Y x − y x, (Y X y x, 1),
X x y − y, (Y X y x, 1),
Y y x − x, (Y X y x, 1),
x^2 − X, (Y X y x, 1),
X^2 − x, (Y X y x, 1),
x y x − y, (Y X y x, 1),
X y − y x, (Y X y x, 1),
X y x − x y, (Y X y x, 1),
x^2 y − y x, (Y X y x, 1),
y X − x y, (Y X y x, 1),
y x y − X, (Y X y x, 1),
Y x y − X, (Y X y x, 1),
Y X − x y, (Y X y x, 1),
[19 Polynomials]

Computing the Reduced Groebner Basis...
...Reduced Groebner Basis Computed.

Here is the Reduced Groebner Basis:
y^2 − 1,
X x − 1,
x X − 1,
Y − y,
x^2 − X,
X^2 − x,
x y x − y,
X y − y x,
y X − x y,
y x y − X,
[10 Polynomials]

Writing Reduced Groebner Basis to Disk... Done.
Writing Involutive Basis to Disk... Done.

0.105u 0.000s 0:00.16 62.5% 197+727k 0+2io 0pf+0w
ma6:mssrc−aux/thesis>
```

Output File:

```
Y; X; y; x;
y^2 − 1; (Y X y x, 1);
X*x − 1; (Y X y x, 1);
x*X − 1; (Y X y x, 1);
Y*y − 1; (Y X y x, 1);
y^2*x − x; (Y X y x, 1);
Y − y; (Y X y x, 1);
Y*x − y*x; (Y X y x, 1);
X*x*y − y; (Y X y x, 1);
Y*y*x − x; (Y X y x, 1);
x^2 − X; (Y X y x, 1);
X^2 − x; (Y X y x, 1);
x*y*x − y; (Y X y x, 1);
```

```
X*y - y*x; (Y X y x, 1);
X*y*x - x*y; (Y X y x, 1);
x^2*y - y*x; (Y X y x, 1);
y*X - x*y; (Y X y x, 1);
y*x*y - X; (Y X y x, 1);
Y*x*y - X; (Y X y x, 1);
Y*X - x*y; (Y X y x, 1);
```

C.1.3 Session 3: Noncommutative Involutive Walks

Task: If $G' := \{y^2 + 2xy,\ y^2 + x^2,\ 5y^3,\ 5xy^2,\ y^2 + 2yx\}$ generates an ideal J over the polynomial ring $\mathbb{Q}\langle x, y\rangle$, compute an Involutive Basis for G' with respect to the left division $\triangleleft$ and the DegRevLex monomial ordering.

Origin of Example: Example 6.2.20 (G' corresponds to a set of initials in the noncommutative Involutive Walk algorithm; we want to compute an Involutive Basis H' for G').

Input File:

```
x; y;
y^2 + 2*x*y;
y^2 + x^2;
5*y^3;
5*x*y^2;
y^2 + 2*y*x;
```

Plan: Apply the program given in Appendix B to the above file, using the '-c2' option to select Algorithm 12 (the DegRevLex monomial ordering and the left division are selected by default).

Program Output:

```
ma6:mssrc-aux/thesis> time involutive -c2 thesis3.in

*** NONCOMMUTATIVE INVOLUTIVE BASIS PROGRAM (GLOBAL DIVISION) ***

Using the DegRevLex Ordering with x > y

Polynomials in the input basis:
y^2 + 2 x y,
y^2 + x^2,
5 y^3,
5 x y^2,
y^2 + 2 y x,
[5 Polynomials]
```

```
Computing an Involutive Basis...
...Involutive Basis Computed.

Here is the Involutive Basis
((Left, Right) Multiplicative Variables in Brackets):
2 y x - x^2, (x y, 1),
y x^2, (x y, 1),
x^3, (x y, 1),
2 x y - x^2, (x y, 1),
y^2 + x^2, (x y, 1),
[5 Polynomials]

Computing the Reduced Groebner Basis...
...Reduced Groebner Basis Computed.

Here is the Reduced Groebner Basis:
2 y x - x^2,
x^3,
2 x y - x^2,
y^2 + x^2,
[4 Polynomials]

Writing Reduced Groebner Basis to Disk... Done.
Writing Involutive Basis to Disk... Done.

0.005u 0.000s 0:00.07 0.0% 0+0k 0+2io 0pf+0w
ma6:mssrc-aux/thesis>
```

More Verbose Program Output: (we select the '-v3' option to obtain more information about the autoreduction that occurs at the start of the algorithm).

```
ma6:mssrc-aux/thesis> time involutive -c2 -v3 thesis3.in

*** NONCOMMUTATIVE INVOLUTIVE BASIS PROGRAM (GLOBAL DIVISION) ***

Using the DegRevLex Ordering with x (AAB) > y (AAA)

Polynomials in the input basis:
AAA^2 + 2 AAB AAA,
AAA^2 + AAB^2,
5 AAA^3,
5 AAB AAA^2,
AAA^2 + 2 AAA AAB,
[5 Polynomials]

Computing an Involutive Basis...
Autoreducing...
Looking at element p = AAA^2 + 2 AAA AAB of basis
Reduced p to AAB AAA - AAA AAB
Looking at element p = 5 AAB AAA^2 of basis
Reduced p to AAB AAA AAB
```

```
Looking at element p = AAB AAA − AAA AAB of basis
Reduced p to AAB AAA − AAA AAB
Looking at element p = 5 AAAˆ3 of basis
Reduced p to AAAˆ2 AAB
Looking at element p = AAB AAA AAB of basis
Reduced p to AAB AAA AAB
Looking at element p = AAB AAA − AAA AAB of basis
Reduced p to AAB AAA − AAA AAB
Looking at element p = AAAˆ2 + AABˆ2 of basis
Reduced p to 2 AAA AAB − AABˆ2
Looking at element p = AAAˆ2 AAB of basis
Reduced p to AAA AABˆ2
Looking at element p = 2 AAA AAB − AABˆ2 of basis
Reduced p to 2 AAA AAB − AABˆ2
Looking at element p = AAB AAA AAB of basis
Reduced p to AABˆ3
Looking at element p = AAA AABˆ2 of basis
Reduced p to AAA AABˆ2
Looking at element p = 2 AAA AAB − AABˆ2 of basis
Reduced p to 2 AAA AAB − AABˆ2
Looking at element p = AAB AAA − AAA AAB of basis
Reduced p to 2 AAB AAA − AABˆ2
Looking at element p = AABˆ3 of basis
Reduced p to AABˆ3
Looking at element p = AAA AABˆ2 of basis
Reduced p to AAA AABˆ2
Looking at element p = 2 AAA AAB − AABˆ2 of basis
Reduced p to 2 AAA AAB − AABˆ2
Looking at element p = AAAˆ2 + 2 AAB AAA of basis
Reduced p to AAAˆ2 + AABˆ2
Looking at element p = 2 AAB AAA − AABˆ2 of basis
Reduced p to 2 AAB AAA − AABˆ2
Looking at element p = AABˆ3 of basis
Reduced p to AABˆ3
Looking at element p = AAA AABˆ2 of basis
Reduced p to AAA AABˆ2
Looking at element p = 2 AAA AAB − AABˆ2 of basis
Reduced p to 2 AAA AAB − AABˆ2
Analysing AAA AAB...
Adding Right Prolongation by variable #0 to S...
Adding Right Prolongation by variable #1 to S...
Analysing AAA AABˆ2...
Adding Right Prolongation by variable #0 to S...
Adding Right Prolongation by variable #1 to S...
Analysing AABˆ3...
Adding Right Prolongation by variable #0 to S...
Adding Right Prolongation by variable #1 to S...
Analysing AAB AAA...
Adding Right Prolongation by variable #0 to S...
Adding Right Prolongation by variable #1 to S...
Analysing AAAˆ2...
Adding Right Prolongation by variable #0 to S...
Adding Right Prolongation by variable #1 to S...
```

```
...Involutive Basis Computed.
Number of Prolongations Considered = 0

Here is the Involutive Basis
((Left, Right) Multiplicative Variables in Brackets):
2 AAA AAB - AAB^2, (all, none),
AAA AAB^2, (all, none),
AAB^3, (all, none),
2 AAB AAA - AAB^2, (all, none),
AAA^2 + AAB^2, (all, none),
[5 Polynomials]

Computing the Reduced Groebner Basis...

Looking at element p = 2 AAA AAB - AAB^2 of basis
Reduced p to 2 AAA AAB - AAB^2

Looking at element p = AAB^3 of basis
Reduced p to AAB^3

Looking at element p = 2 AAB AAA - AAB^2 of basis
Reduced p to 2 AAB AAA - AAB^2

Looking at element p = AAA^2 + AAB^2 of basis
Reduced p to AAA^2 + AAB^2
Number of Reductions Carried out = 34
...Reduced Groebner Basis Computed.

Here is the Reduced Groebner Basis:
2 AAA AAB - AAB^2,
AAB^3,
2 AAB AAA - AAB^2,
AAA^2 + AAB^2,
[4 Polynomials]

Writing Reduced Groebner Basis to Disk... Done.
Writing Involutive Basis to Disk... Done.

0.000u 0.005s 0:00.04 0.0% 0+0k 0+2io 0pf+0w
ma6:mssrc-aux/thesis>
```

Output File:

```
x; y;
2*y*x - x^2; (x y, 1);
y*x^2; (x y, 1);
x^3; (x y, 1);
2*x*y - x^2; (x y, 1);
y^2 + x^2; (x y, 1);
```

C.1.4 Session 4: Ideal Membership

Task: If $F := \{x+y+z-3,\ x^2+y^2+z^2-9,\ x^3+y^3+z^3-24\}$ generates an ideal J over the polynomial ring $\mathbb{Q}\langle x,y,z\rangle$, are the polynomials $x+y+z-3$; $x+y+z-2$; xz^2+yz^2-1; $zyx+1$ and x^{10} members of J?

Input File:

```
x; y; z;
x + y + z − 3;
xˆ2 + yˆ2 + zˆ2 − 9;
xˆ3 + yˆ3 + zˆ3 − 24;
```

Plan: To solve the ideal membership problem for the five given polynomials, we first need to obtain a Gröbner or Involutive Basis for F. We shall do this by applying the program given in Appendix B to compute an Involutive Basis for F with respect to the DegLex monomial ordering and the right division $\rhd$ (this requires the '-d' and '-s4' options respectively). Once the Involutive Basis has been computed (which then allows the program to compute the unique reduced Gröbner Basis G for F), we can start an ideal membership problem solver (courtesy of the '-p' option) which allows us to type in a polynomial p and find out whether or not p is a member of J (the program reduces p with respect to G, testing to see whether or not a zero remainder is obtained).

Program Output:

```
ma6:mssrc−aux/thesis> involutive −c2 −d −p −s4 thesis4.in

*** NONCOMMUTATIVE INVOLUTIVE BASIS PROGRAM (GLOBAL DIVISION) ***

Using the DegLex Ordering with x > y > z

Polynomials in the input basis:
x + y + z − 3,
xˆ2 + yˆ2 + zˆ2 − 9,
xˆ3 + yˆ3 + zˆ3 − 24,
[3 Polynomials]

Computing an Involutive Basis...
Added Polynomial #4 to Basis...
Added Polynomial #5 to Basis...
Added Polynomial #6 to Basis...
Added Polynomial #7 to Basis...
Added Polynomial #8 to Basis...
Added Polynomial #9 to Basis...
Added Polynomial #10 to Basis...
Added Polynomial #11 to Basis...
```

```
Added Polynomial #12 to Basis...
Added Polynomial #13 to Basis...
Added Polynomial #14 to Basis...
Added Polynomial #15 to Basis...
Added Polynomial #16 to Basis...
Added Polynomial #17 to Basis...
Added Polynomial #18 to Basis...
Added Polynomial #19 to Basis...
Added Polynomial #20 to Basis...
Added Polynomial #21 to Basis...
Autoreduction reduced the basis to size 13...
...Involutive Basis Computed.

Here is the Involutive Basis
((Left, Right) Multiplicative Variables in Brackets):
x + y + z - 3, (1, x y z),
z x + z y + z^2 - 3 z, (1, x y z),
y z - z y, (1, x y z),
z^3 - 3 z^2 + 1, (1, x y z),
z^2 y^2 - y - z, (1, x y z),
z^2 y x + z, (1, x y z),
z^2 y z - 3 z^2 y + y, (1, x y z),
z y z - z^2 y, (1, x y z),
z y x + 1, (1, x y z),
z y^2 + z^2 y - 3 z y - 1, (1, x y z),
z^2 x + z^2 y - 1, (1, x y z),
y x - z^2 + 3 z, (1, x y z),
y^2 + z y + z^2 - 3 y - 3 z, (1, x y z),
[13 Polynomials]

Computing the Reduced Groebner Basis...
...Reduced Groebner Basis Computed.

Here is the Reduced Groebner Basis:
x + y + z - 3,
y z - z y,
z^3 - 3 z^2 + 1,
y^2 + z y + z^2 - 3 y - 3 z,
[4 Polynomials]

Writing Reduced Groebner Basis to Disk... Done.
Writing Involutive Basis to Disk... Done.

*** IDEAL MEMBERSHIP PROBLEM SOLVER ***

Source: Disk (d) or Keyboard (k)? ...k

Please enter a polynomial (e.g. x*y^2-z)
(A semicolon terminates the program)...x+y+z-3
Polynomial x + y + z - 3 IS a member of the ideal.
Please enter a polynomial (e.g. x*y^2-z)
(A semicolon terminates the program)...x+y+z-2
Polynomial y + 2 z - 2 is NOT a member of the ideal.
```

```
Please enter a polynomial (e.g. x*y^2-z)
(A semicolon terminates the program)...x*z^2+y*z^2-1
Polynomial x z^2 + y z^2 - 1 IS a member of the ideal.
Please enter a polynomial (e.g. x*y^2-z)
(A semicolon terminates the program)...z*y*x+1
Polynomial z y x + 1 IS a member of the ideal.
Please enter a polynomial (e.g. x*y^2-z)
(A semicolon terminates the program)...x^10
Polynomial x^10 is NOT a member of the ideal.
Please enter a polynomial (e.g. x*y^2-z)
(A semicolon terminates the program)...;
ma6:mssrc-aux/thesis>
```

Output File:

```
x; y; z;
x + y + z - 3; (1, x y z);
z*x + z*y + z^2 - 3*z; (1, x y z);
y*z - z*y; (1, x y z);
z^3 - 3*z^2 + 1; (1, x y z);
z^2*y^2 - y - z; (1, x y z);
z^2*y*x + z; (1, x y z);
z^2*y*z - 3*z^2*y + y; (1, x y z);
z*y*z - z^2*y; (1, x y z);
z*y*x + 1; (1, x y z);
z*y^2 + z^2*y - 3*z*y - 1; (1, x y z);
z^2*x + z^2*y - 1; (1, x y z);
y*x - z^2 + 3*z; (1, x y z);
y^2 + z*y + z^2 - 3*y - 3*z; (1, x y z);
```

Bibliography

[1] B. Amrhein, O. Gloor and W. Küchlin. On the Walk. *Theoret. Comput. Sci.* **187 (1-2)** (1997) 179–202.
URL http://dx.doi.org/10.1016/S0304-3975(97)00064-9

[2] J. Apel. A Gröbner Approach to Involutive Bases. *J. Symbolic Comput.* **19 (5)** (1995) 441–457.
URL http://dx.doi.org/10.1006/jsco.1995.1026

[3] J. Apel. The Computation of Gröbner Bases Using an Alternative Algorithm. *Progr. Comput. Sci. Appl. Logic* **15** (1998) 35–45.

[4] J. Apel. The Theory of Involutive Divisions and an Application to Hilbert Function Computations. *J. Symbolic Comput.* **25 (6)** (1998) 683–704.
URL http://dx.doi.org/10.1006/jsco.1997.0194

[5] T. Baader and N. Tobias. *Term Rewriting and All That*. Cambridge University Press (1998).

[6] J. Backelin, S. Cojocaru and V. Ufnarovski. Bergman 1.0 User Manual (1998).
URL http://servus.math.su.se/bergman/

[7] T. Becker and V. Weispfenning. *Gröbner Bases: A Commutative Approach to Commutative Algebra*. Springer-Verlag (1993).

[8] G. M. Bergman. The Diamond Lemma for Ring Theory. *Adv. in Math.* **29 (2)** (1978) 178–218.
URL http://dx.doi.org/10.1016/0001-8708(78)90010-5

[9] W. Böge, R. Gebauer and H. Kredel. Some Examples for Solving Systems of Algebraic Equations by Calculating Groebner Bases. *J. Symbolic Comput.* **2 (1)** (1986) 83–98.

[10] B. Buchberger. A Criterion for Detecting Unnecessary Reductions in the Construction of Gröbner Bases. In *Symbolic and Algebraic Computation (EUROSAM '79, Int. Symp., Marseille, 1979)*, volume 72 of *Lecture Notes in Comput. Sci.*, 3–21. Springer, Berlin (1979).

[11] B. Buchberger. An Algorithmic Criterion for the Solvability of a System of Algebraic Equations. *Translation of PhD thesis by M. Abramson and R. Lumbert*. In B. Buchberger and F. Winkler, editors, *Gröbner Bases and Applications*, volume 251 of *Proc. London Math. Soc.*, 535–545. Cambridge University Press (1998).

[12] B. Buchberger and F. Winkler, editors. *Gröbner Bases and Applications*, volume 251 of *Proc. London Math. Soc.*. Cambridge University Press (1998).

[13] J. Calmet, M. Hausdorf and W. M. Seiler. A Constructive Introduction to Involution. In R. Akerkar, editor, *Proc. Int. Symp. Applications of Computer Algebra – ISACA 2000*, 33–50. Allied Publishers, New Delhi (2001).

[14] J. A. Camberos, L. A. Lambe and R. Luczak. Hybrid Symbolic-Numeric Methodology for Fluid Dynamic Simulations. In *34th AIAA Fluid Dynamics Conference and Exhibit*, volume 2004-2330. Portland, OR (2004).

[15] J. A. Camberos, L. A. Lambe and R. Luczak. Computational Physics with Hybrid Symbolic-Numeric Methodology with Multidisciplinary Applications. *Department of Defense HPC User's Group Meeting* (2005).

[16] J. A. Camberos, L. A. Lambe and R. Luczak. Hybrid Symbolic-Numeric Methodology: Views and Visions. In *43rd AIAA Aerospace Sciences Meeting*, volume 2005-0092. Reno, NV (2005).

[17] J. F. Carlson. Cohomology, Computations, and Commutative Algebra. *Notices Amer. Math. Soc.* **52 (4)** (2005) 426–434.

[18] M. Collart, M. Kalkbrener and D. Mall. Converting Bases with the Gröbner Walk. *J. Symbolic Comput.* **24 (3-4)** (1997) 465–469.
URL http://dx.doi.org/10.1006/jsco.1996.0145

[19] N. Dershowitz. A Taste of Rewrite Systems. In P. E. Lauer, editor, *Functional Programming, Concurrency, Simulation and Automated Reasoning*, volume 693 of *Lecture Notes in Comput. Sci.*, 199–228. Springer (1993).

[20] G. A. Evans. Noncommutative Involutive Bases. In Q.-N. Tran, editor, *Proc. 10th Int. Conf. Applications of Computer Algebra*, 49–64. Beaumont, Texas, USA (2004).

[21] J. C. Faugère, P. Gianni, D. Lazard and T. Mora. Efficient Computation of Zero-dimensional Gröbner Bases by Change of Ordering. *J. Symbolic Comput.* **16 (4)** (1993) 329–344.
URL http://dx.doi.org/10.1006/jsco.1993.1051

[22] R. Fröberg. *An Introduction to Gröbner Bases*. John Wiley & Sons (1998).

[23] V. P. Gerdt. Involutive Division Technique: Some Generalizations and Optimizations. *J. Math. Sci. (N. Y.)* **108 (6)** (2002) 1034–1051.
URL http://dx.doi.org/10.1023/A:1013596522989

[24] V. P. Gerdt. Involutive Algorithms for Computing Gröbner Bases. In S. Cojocaru, G. Pfister and V. Ufnarovski, editors, *Computational Commutative and Non-Commutative Algebraic Geometry*, volume 196 of *NATO Science Series: Computer and Systems Sciences*, 199–225. IOS Press (2005).

[25] V. P. Gerdt and Yu. A. Blinkov. Involutive Bases of Polynomial Ideals. *Math. Comput. Simulation* **45 (5-6)** (1998) 519–541.
URL http://dx.doi.org/10.1016/S0378-4754(97)00127-4

[26] V. P. Gerdt and Yu. A. Blinkov. Minimal Involutive Bases. *Math. Comput. Simulation* **45 (5-6)** (1998) 543–560.
URL http://dx.doi.org/10.1016/S0378-4754(97)00128-6

[27] V. P. Gerdt, Yu. A. Blinkov and D. A. Yanovich. Construction of Janet Bases I. Monomial Bases. In V. G. Ghanzha, E. W. Mayr and E. V. Vorozhtsov, editors, *Computer Algebra in Scientific Computing (CASC) 2001*, 233–247. Springer-Verlag, Berlin (2001).

[28] V. P. Gerdt, Yu. A. Blinkov and D. A. Yanovich. Construction of Janet Bases II. Polynomial Bases. In V. G. Ghanzha, E. W. Mayr and E. V. Vorozhtsov, editors, *Computer Algebra in Scientific Computing (CASC) 2001*, 249–263. Springer-Verlag, Berlin (2001).

[29] A. Giovini, T. Mora, G. Niedi, L. Robbiano and C. Traverso. "One sugar cube, please" OR Selection Strategies in Buchberger Algorithm. In *ISSAC '91: Proc. Int.*

Symp. Symbolic and Algebraic Computation, 49–54. ACM Press, New York (1991).
URL http://dx.doi.org/10.1145/120694.120701

[30] O. D. Golubitsky. Involutive Gröbner Walk. *Fundam. Prikl. Mat.* **7 (4)** (2001) 993–1001.

[31] G. M. Greuel, G. Pfister and H. Schönemann. SINGULAR 3.0.0. A Computer Algebra System for Polynomial Computations, Centre for Computer Algebra, University of Kaiserslautern (2005).
URL http://www.singular.uni-kl.de

[32] J. W. Helton and M. Stankus. NCGB 3.1: NonCommutative Gröbner Basis Package (2002).
URL http://math.ucsd.edu/~ncalg/

[33] A. Heyworth. Rewriting as a special case of Gröbner Basis Theory. In M. Atkinson, N. D. Gilbert, J. Howie, S. A. Linton and E. F. Robertson, editors, *Computational and Geometric Aspects of Modern Algebra (Edinburgh, 1998)*, volume 275 of *Proc. London Math. Soc.*, 101–105. Cambridge University Press (2000).

[34] F. Hreinsdóttir. A case where choosing a product order makes the calculation of a Gröbner Basis much faster. *J. Symbolic Comput.* **18 (4)** (1994) 373–378.
URL http://dx.doi.org/10.1006/jsco.1994.1053

[35] M. Janet. *Lecons sur les systémes d'équations aux deriveés partielles*. Gauthier-Villars, Paris (1929).

[36] J.-P. Jouannaud. Rewrite Proofs and Computations. In H. Schwichtenberg, editor, *Proof and Computation*, volume 139 of *Computer and Systems Sciences*. Springer-Verlag (1995).

[37] B. J. Keller. Algorithms and Orders for Finding Noncommutative Gröbner Bases. PhD thesis, Virginia Tech (1997).

[38] B. W. Kernighan and D. M. Ritchie. *The C Programming Language (2nd Edition)*. Prentice Hall (1988).

[39] D. E. Knuth and P. B. Bendix. Simple word problems in universal algebras. In J. Leech, editor, *Computational Problems in Abstract Algebra*, 263–297. Pergamon Press (1970).

[40] L. A. Lambe. FAlg (An ANSI C Library for Free Associative Algebra Calculations): User Guide (2001).

[41] L. A. Lambe. FMon (An ANSI C Library for Free Monoid Calculations): User Guide (2001).

[42] L. A. Lambe. Letter of Support for Research at the University of Wales, Bangor (2005).

[43] L. A. Lambe, R. Luczak and J. Nehrbass. A new finite difference method for the Helmholtz equation using symbolic computation. *Int. J. Comput. Eng. Sci.* **4 (1)** (2003) 121–144.
URL http://dx.doi.org/10.1142/S1465876303000739

[44] E. Mansfield. Differential Gröbner Bases. PhD thesis, University of Sydney (1991).

[45] T. Mora. Gröbner Bases for non-commutative polynomial rings. In J. Calmet, editor, *AAECC-3: Proc. 3rd Int. Conf. on Algebraic Algorithms and Error-Correcting Codes (Grenoble, France, July 15–19, 1985)*, volume 223 of *Lecture Notes in Comput. Sci.*, 353–362. Springer (1986).

[46] MSSRC (Multidisciplinary Software Systems Research Corporation).
URL http://www.mssrc.com

[47] J. F. Pommaret. *Systems of Partial Differential Equations and Lie Pseudogroups*. Gordon and Breach Science Publishers (1978).

[48] L. Robbiano. Term Orderings on the Polynomial Ring. In *EUROCAL '85: Research Contributions from the European Conference on Computer Algebra – Volume 2*, volume 204 of *Lecture Notes in Comput. Sci.*, 513–517. Springer (1985).

[49] SciFace Software. MuPAD Version 3.1. Paderborn, Germany (2005).
URL http://www.mupad.com/

[50] W. M. Seiler. A Combinatorial Approach to Involution and δ-Regularity I: Involutive Bases in Polynomial Algebras of Solvable Type (2002). Preprint Universitä Mannheim.

[51] W. M. Seiler. A Combinatorial Approach to Involution and δ-Regularity II: Structure Analysis of Polynomial Modules with Pommaret Bases (2002). Preprint Universitä Mannheim.

[52] J. M. Thomas. *Differential Systems*. AMS, New York (1937).

[53] Q.-N. Tran. A Fast Algorithm for Gröbner Basis Conversion and its Applications. *J. Symbolic Comput.* **30 (4)** (2000) 451–467.
URL http://dx.doi.org/10.1006/jsco.1999.0416

[54] A. Wachowski and L. Wachowski. The Matrix. Film (Warner Bros.) (1999).

[55] Waterloo Maple Inc. Maple Version 10. Waterloo, Ontario, Canada. (2005).
URL http://www.maplesoft.com

[56] V. Weispfenning. Admissible orders and linear forms. *ACM Sigsam Bull.* **21 (2)** (1987) 16–18.
URL http://doi.acm.org/10.1145/24554.24557

[57] Wolfram Research, Inc. Mathematica Version 5.1. Champaign, Illinois (2004).
URL http://www.wolfram.com/

[58] A. Yu. Zharkov and Yu. A. Blinkov. Involution Approach to Investigating Polynomial Systems. *Math. Comput. Simulation* **42 (4-6)** (1996) 323–332.
URL http://dx.doi.org/10.1016/S0378-4754(96)00006-7

Index

T

U

W